Bifurcations of Planar Random Differential Equations with Bounded Noise

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Bounded Noise in Discrete Time Dynamical Systems

M - a manifold. $\Delta = B_\epsilon(0) \in \mathbb{R}^n$
$F : M \times \Delta \to M$ - a discrete dynamical system
ν - some a.c. probability measure on Δ.
ν^∞ - product measure on Δ^∞.

For $\bar{t} \in \Delta^k$, $F^k_{\bar{t}} = F_{t_k} \circ F_{t_{k-1}} \circ \cdots \circ F_{t_1}(x)$.

This is equivalent under the right assumptions to a Stochastic Process with transition function:

$$P(x, A) = \nu(\{t \in \Delta : F_t(x) \in A\}).$$

Random differential equations (RDEs)

$$\dot{x} = f_\lambda(x, \xi_t) \qquad x \in \mathbb{R}^n \tag{1}$$

Parameter $\lambda \in \mathbb{R}$ is varied

ξ_t will be a realization of some noise process.

ξ_t takes values in a closed disk $\Delta \subset \mathbb{R}^n$.

$f_\lambda(x, \Delta)$ is a convex set for each $x \in X$.

$f_\lambda(x, v)$ is a smooth vector field depending smoothly on $\lambda \in \mathbb{R}$ and $v \in \Delta$.

The Noise ξ_t

$\xi_t \in \mathcal{U} = \{\xi : \mathbb{R} \to \Delta, \xi$ measurable$\}$.

The flow defined by the shift:

$$\theta^t : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \qquad \theta^t(\xi_s) = \xi_{s+t},$$

is then a continuous dynamical system with the weak topology on $\mathcal{U}$.

Existence, Uniqueness, Smooth Dependence

Since $\xi \in \mathcal{U}$ is measurable, and f is smooth and bounded, the differential equation (1) has unique, global solutions $\Phi_\lambda^t(x,\xi)$ (in the sense of Caratheodory), i.e.:

$$\Phi_\lambda^t(x,\xi) = x + \int_0^t f(\Phi_\lambda^t(x,\xi), \xi_s)\, ds,$$

for any $\xi \in \mathcal{U}$ and all initial conditions x in X, and the solutions are absolutely continuous in t.

Solutions depend smoothly on λ and continuously on ξ in the space $\mathcal{U}$.

Minimal Forward Invariant Sets

A set $F \subset X$ is *forward invariant* if

$$\Phi_\lambda^t(F,\mathcal{U}) \subset F \tag{2}$$

for all $t > 0$.

Denote by $\mathcal{F}$ the collection of forward invariant sets.

There is a partial ordering on $\mathcal{F}$ by inclusion, i.e. $E \prec F$ if $E \subset F$.

We call $E \subset \mathcal{F}$ a *minimal forward invariant* (MFI) set if it is minimal with respect to the partial ordering $\prec$.

MFI sets are forward orbits

Denote

$$\Phi_\lambda^t(x,U) = \{y \in X \mid y = \Phi_\lambda^t(x,\xi) \text{ for some } \xi \in U \text{ and } t > 0\}$$

H2. There exist $r_0 > 0$ and $t_1 > 0$ such that for each $x \in X$,

$$\Phi_\lambda^t(x,\mathcal{U}) \supset B(\Phi_\lambda^t(x,0), r_0) \qquad \forall t > t_1.$$

Theorem 1 (Doob '53, H&Y '06) *Under assumption* **H2** *MFI sets exist. They open and connected. The closures of distinct MFI sets are disjoint. If x is any point in an MFI set E, then E is equal to the forward orbit of x, i.e.*

$$E = O^+(x) \equiv \bigcup_{t>0} \Phi_\lambda^t(x,\mathcal{U}).$$

1-d MFI sets, $\dot{x} = f(x) + \xi(t)$

Let: $f^{+}(x) = f(x) + \max(\xi)$, $f^{-}(x) = f(x) + \min(\xi)$

Proposition 2 *If* (a, b) *is an MFI set, then* $0 \in f(x, \Delta), \forall x \in (a, b)$*.*

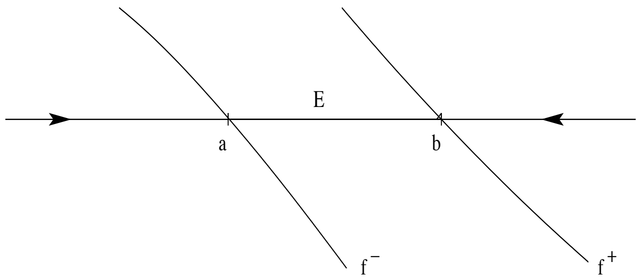
Proposition 3 *If* (a, b) *is an MFI set then*

$$f(a, \xi) \geq 0 \quad \text{and} \quad f(b, \xi) \leq 0 \tag{3}$$

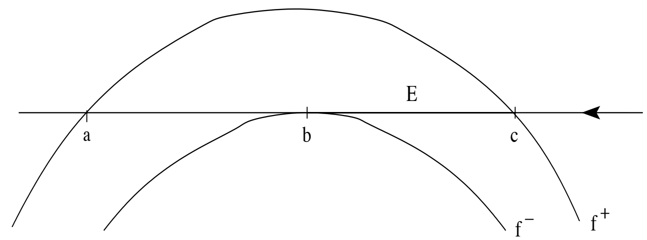
for all $\xi \in \Delta = [-\epsilon, \epsilon]$ *and that* $f_{-}(a) = 0$ *and* $f_{+}(b) = 0$*. Further,* $f'_{-}(a) \leq 0$ *and* $f'_{+}(b) \leq 0$*.*

Examples of MFI sets in 1-d

(a)

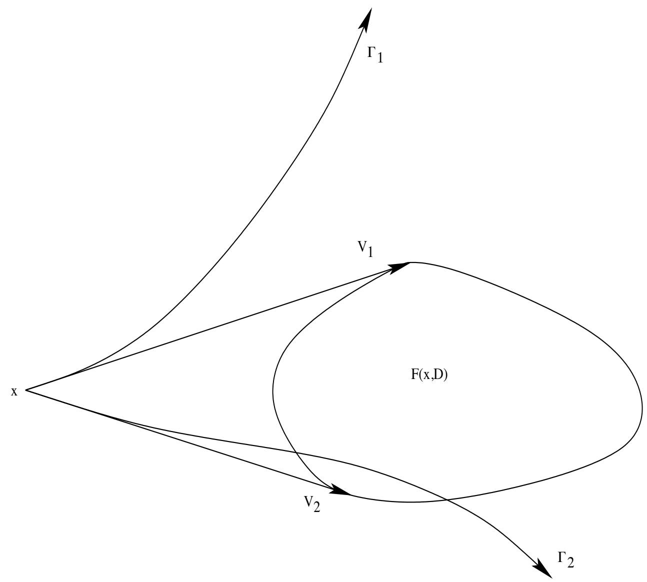


(b)



(a) A stable one dimensional MFI set. Both endpoints of $E = (a, b)$ are hyperbolic. (b) A random saddle-node in one dimension. $E = (b, c)$ is minimal forward invariant.

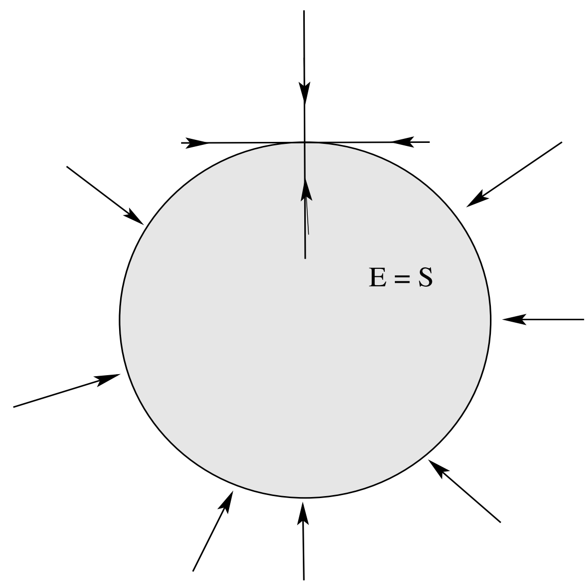
$\mathbb{R}^2$ - extreme vector fields and curves



Lemma 4 *MFI sets are bounded by extremal curves.*

Example. Perturbed linear stable node with repeated eigenvalues but distinct eigenvectors

$$\dot{x} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \xi_t, \qquad |\xi_t| \leq \epsilon.$$

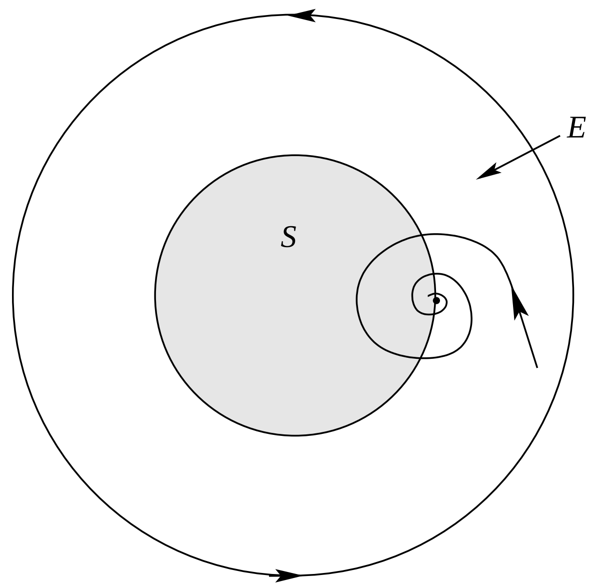


$S = \{x : 0 \in f(x, \Delta)\}$ – singular set.

In this case the MFI set E is S.

Example. Perturbed linear stable focus

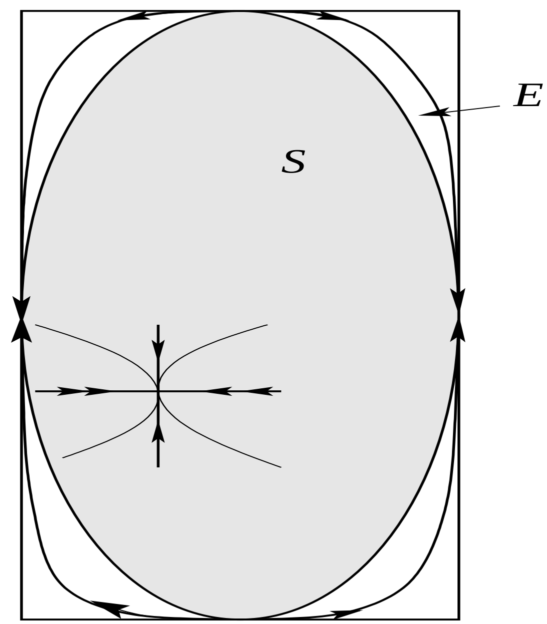
$$\dot{x} = -\begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix} x + \xi_t$$



Example. Perturbed linear stable node with distinct eigenvalues

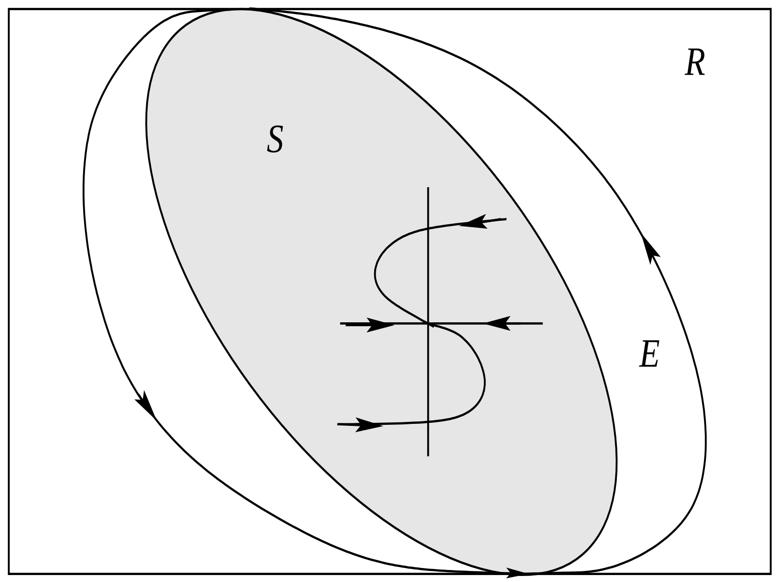
$$\dot{x} = -\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} x + \xi_t$$

$0 < a < 1$



Example. Perturbed stable node with a single eigenvector

$$\dot{x} = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \xi_t$$



MFI sets and Attractors

Consider: $\dot{x} = f(x, \epsilon\xi_t)$ - a perturbation of an ODE.

Definition 5 *A set A is called an* attractor *for $\epsilon = 0$ if it is:*

A1 *Invariant and compact.*

A2 *There is a neighborhood U of A such that for all $x \in \overline{U}$, $\Phi^t(x, 0) \in U$ for all $t \geq 0$ and $\Phi^t(x, 0) \to A$ as $t \to \infty$.*

A3 *There is $x \in U$ s.t. A is the ω limit set of x, or, A contains a point with a dense orbit, or, A is chain transitive.*

If A satisfies A1 and A2 only, it is said to be asymptotically attracting *or an* attracting set*. We call $\overline{U}$ a* trapping region*.*

MFI sets and Attractors

Consider:

$$\dot{x} = f(x, \epsilon\xi_t) \tag{4}$$

where ϵ is small. For $\epsilon = 0$ the system is deterministic.

Theorem 6 *Suppose that (4) satisfies H2 and that for $\epsilon = 0$ it has an attractor A. Then for ϵ sufficiently small, (4) has a MFI set that is a small neighborhood of A. Suppose that U is a trapping region for A, then given ϵ small enough, this MFI set is unique in U.*

MFI sets and Attractors

Corollary 7 *Suppose (4) satisfies H1 and H2. If* x_0 *is an asymptotically stable equilibrium for* $\epsilon = 0$*, then for all sufficiently small* $\epsilon > 0$ *the system has a small MFI set that contains* x_0*. If* Γ *is an asymptotically stable limit cycle for* $\epsilon = 0$*, then for small* $\epsilon > 0$ *the system has a MFI set that is a small tubular neighborhood of* Γ*.*

Probability on the Noise Space

Suppose $\exists$ a θ^t-invariant, ergodic probability measure $\mathcal{P}$ on $\mathcal{U}$,

The flow generates a stochastic process with transition probabilities given by

$$\begin{aligned} P^t_\lambda(x, B) &:= \mathcal{P}(\{\xi \in \mathcal{U} : \Phi^t_\lambda(x, \xi) \in B\}) \\ &= \int_{\{\xi : \Phi^t_\lambda(x,\xi) \in B\}} d\mathcal{P}(\xi), \end{aligned} \tag{5}$$

for any Borel set B.

Generally, the process cannot satisfy the Markov condition. However, under mild conditions the process defined by the pair (P^t_λ, ξ_t) is Markov.

Assumption on the noise

H3. There exists $t_2 > 0$ so that the push forward of $\mathcal{P}$ via $\Phi^t_\lambda(x, \cdot)$ is equivalent to a Riemannian measure m on $\Phi^t_\lambda(x,\mathcal{U})$ for all $t > t_2$ and all $x \in X$.

A probability μ on X is *stationary* if

$$\iint \phi(y)\, d_y P^t_\lambda(x,y) d\mu(x) = \int \phi\, d\mu,$$

for all $\phi \in C(X,\mathbb{R})$.

A stationary measure μ is called *ergodic* if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(\Phi^t_\lambda(x,\xi))dt = \int \phi\, d\mu \tag{6}$$

for $\mu \times \mathcal{P}$-a.e. $(x,\xi) \in X \times \mathcal{U}$ and all $\phi \in C(X,\mathbb{R})$.

Stationary measures are supported on MFI sets

Theorem 8 (Doob '53, H&Y '06) *Let (1) have bounded noise on a smooth, compact manifold X whose flow satisfies* **H2** *and* **H3**. *Then there are a finite number of ergodic, physical, invariant probability measures $\mu_1, \ldots, \mu_k$ on X. Each μ_i is supported on the closure of an MFI set E_i. Further, given any $x \in X$ and almost any $\xi \in \mathcal{U}$, there exists $t^* = t^*(x, \xi)$, such that $\Phi_\lambda^t(x, \xi) \in E_i$ for some i and all $t > t^*$.*

Bifurcation of MFI sets

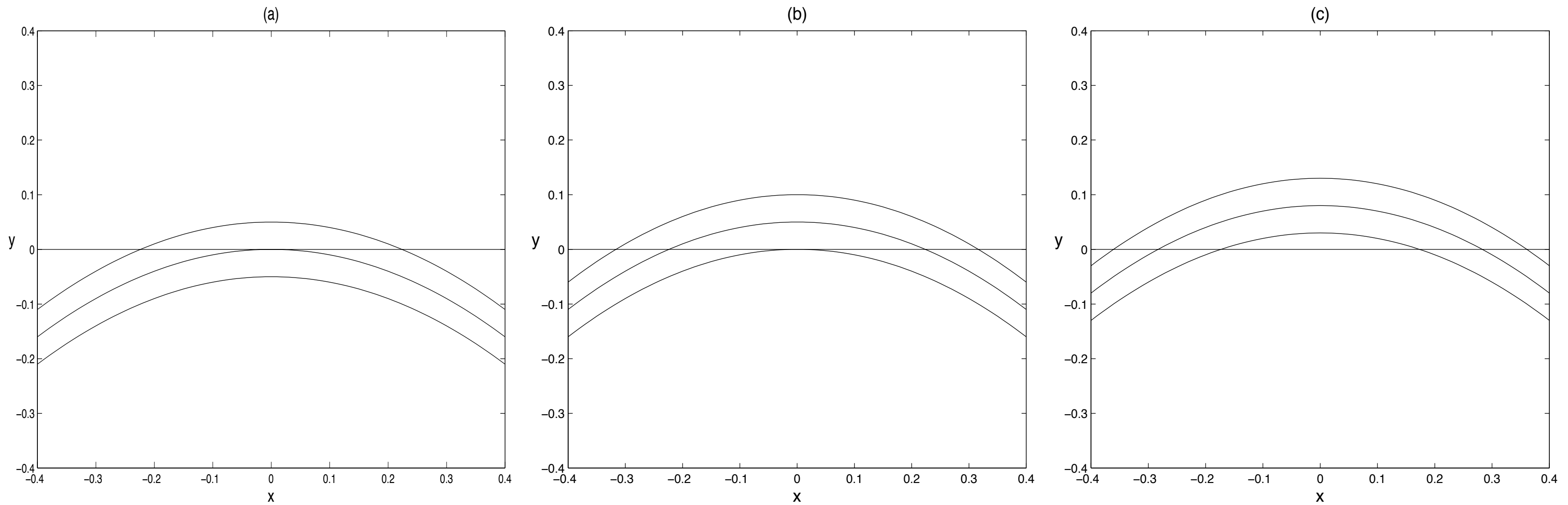
Definition 9 *A* bifurcation *of MFI sets is said to occur in a parameterized family of random differential equations if either:*

B1 *The number of MFI sets changes.*

B2 *An MFI set changes discontinuously with respect to the Hausdorff metric.*

Example. Saddle-node bifurcation in 1-d

$$\dot{x} = F(x) = a - x^2 \xi_t, \qquad |\xi_t| \le \epsilon.$$



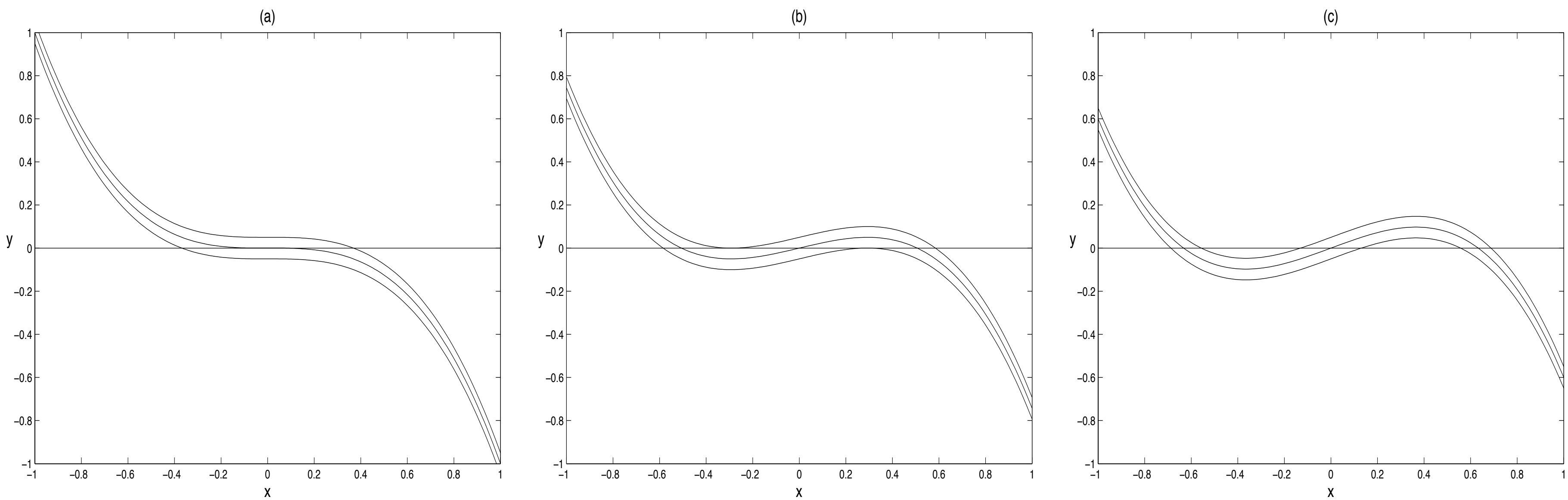
(a) The deterministic system has a saddle-node bifurcation at $a = 0$. (b) The stochastic bifurcation occurs at $a = \epsilon$. (c) For $a \ge \epsilon$ the flow has a trapping interval in a neighborhood of the stable node of $F(x)$.

Saddle-nodes in 1-d

Theorem 10 *The saddle-node is the only co-dimension one bifurcation in one dimensional RDE with bounded noise without symmetries.*

Pitchfork bifurcation

$$\dot{x} = ax - x^3 + \xi_t, \qquad |\xi_t| \leq \epsilon.$$

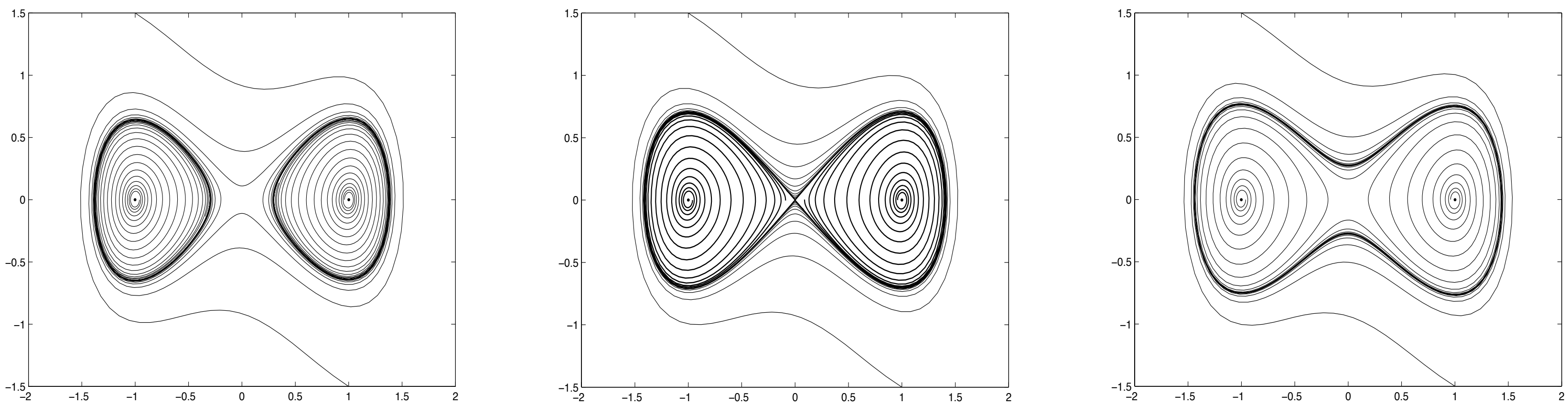


(a) There is one large MFI set. (b) Pitchfork bifurcation. (c) There are two small MFI sets for all $a \geq a^*$

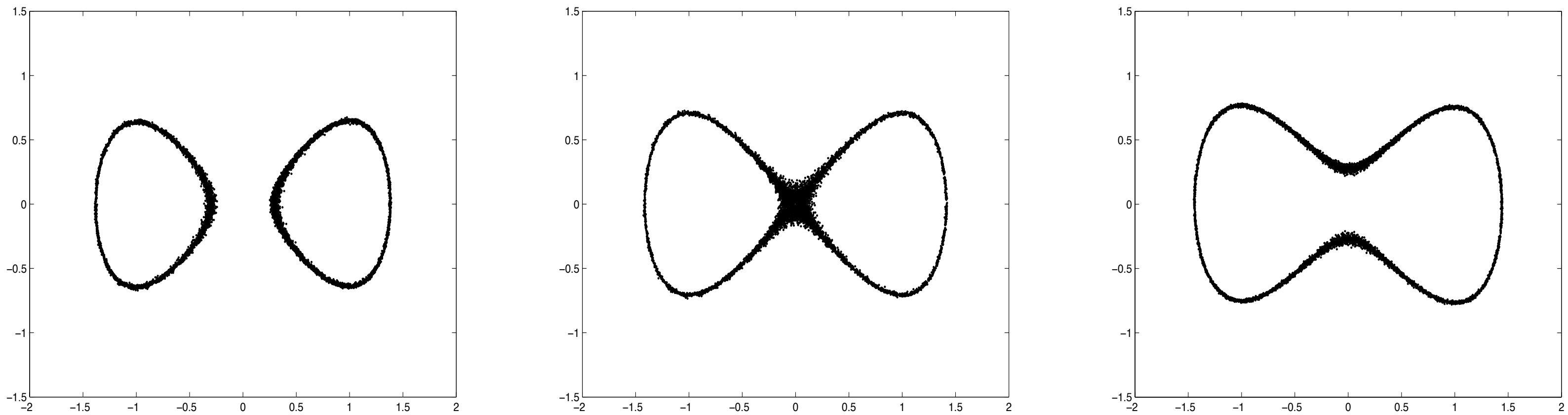
Stable homoclinic bifurcation in planar flows

$$X(x,y) := y \qquad \dot{x} = X - aY$$
$$Y(x,y) := x - x^3 + \delta H(x,y)y \qquad \dot{y} = aX + Y$$

Deterministic homoclinic bifurcation:



Random homoclinic bifurcation



Phase portraits with $\delta = .6$ and added noise with level $\epsilon = .8$. In the first plot ($a = -.01$) there is a pair of disjoint invariant densities. In the second plot ($a = 0$), there is a single invariant density. In the third plot ($a = .01$), the support of the invariant density has undergone a topological change.

Definition 11 *We will denote by* R^∞ *the space of bounded noise vector fields* f *satisfying* **H1** *and* **H2**. *We will take as a topology on* R^∞ *the* C^∞ *topology on the vector fields* $f : X \times \mathbf{\Delta} \to TX$.

Definition 12 *We will say that an MFI set* E *for* f *is* stable *if there is a neighborhood* $U \supset E$ *such that if* $\tilde{f}$ *is sufficiently close to* f *in* R^∞ *then* $\tilde{f}$ *has exactly one MFI set* $\tilde{E} \subset U$ *and* $\tilde{E}$ *is close to* E *in the Hausdorff metric. We will say that* $f \in R^\infty$ *is* stable *if all of its MFI sets* $\{E_i\}$ *are stable.*

Definition 13 *We say that an MFI set* E *for* (1) *is* isolated *if for any proper neighborhood* U *(*$\overline{E} \subset U$*) there is an open forward invariant set* $F \subset U$ *such that* $\overline{E} \subset F$*,* F *contains no other MFI set and* $\overline{\Phi^t_\lambda(F,\mathcal{U})} \subset F$ *for all* $t > 0$*. Such an* F *is called an* isolating set *for* E*.*

Proposition 14 *Isolated MFI sets are stable.*

S. W. Lamb, M. Rasmussen, and C. S. Rodrigues '11

In the context of continuous maps of a metric space:

Theorem 15 *Discontinuous changes in the collection of minimal invariant sets occur either as* **explosions** *or as* **instantaneous appearances***.*

Bounded Noise in Cell Cycle Models

There are two types of natural bounded noise in yeast culture:

- *Asymmetrical division*
- *Epigenetic/physiological differences*

Mother cells are bigger at division and take less time to mature.

Every individual cell has unique structure that effect growth and division.

Cell-cycle models with noise

Stochastic Differential Equation:

$$dx_i = a(x_i, I) + \sigma\, dW_i = \begin{Bmatrix} dt, & \text{if} \quad x_i \notin R, \\ (1 + f(I))dt, & \text{if} \quad x_i \in R \end{Bmatrix} + \sigma\, dW_i.$$

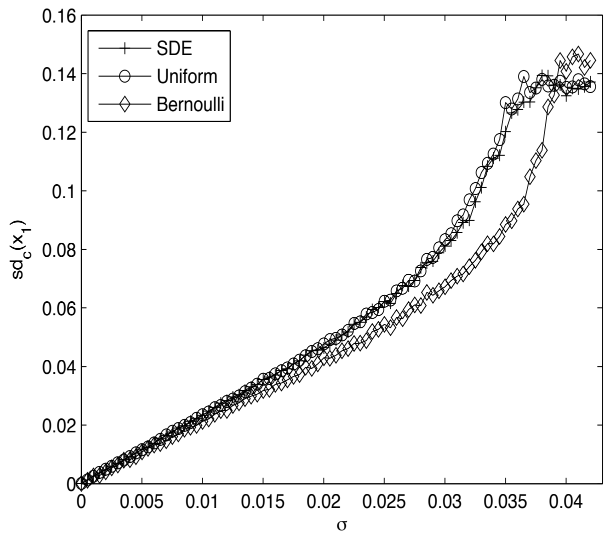
Random Differential Equation:

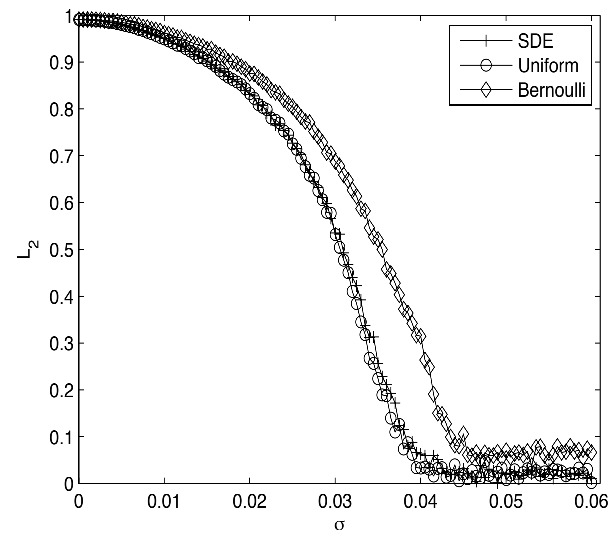
$$\frac{dx_i}{dt} = a(x_i, I) + U_i = \begin{Bmatrix} 1, \quad \text{if} \quad x_i \notin R \\ 1 + f(I), \quad \text{if} \quad x_i \in R, \end{Bmatrix} + U_i$$

where

$$I(\bar{x}) \equiv \frac{\#\{j : x_j \in S\}}{n}, \tag{7}$$

$U_i(t)$ a random variable that changes at each division.

(a) 

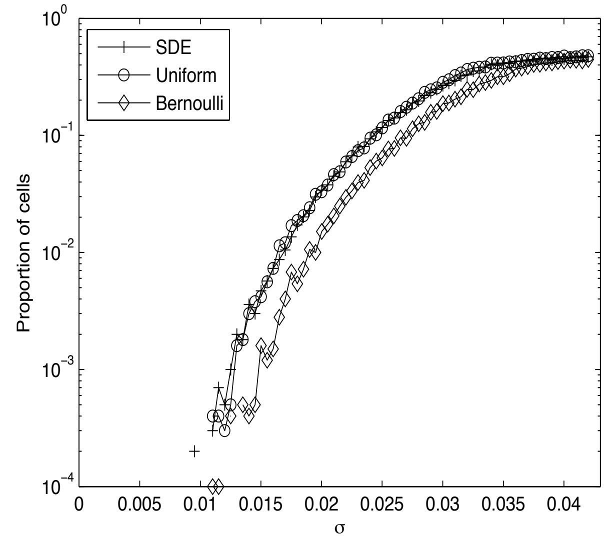
(b) 

(a) The circular standard deviations of cells in the first cluster for the three models at different noise levels.

(b) L_2 the total second Fourier coefficient on the distribution.

There is no difference in the statistics of the three models.

Bifurcations?



Semilogy plot for the number of cells have escaped from their initial clusters for the three models.

There is no essential difference in the three models.

The Hard Bifurcation can be calculated. It happens for extremely small noise.