Bifurcations of Random Differential Equations with Bounded Noise

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Assumptions on the RDE

Random differential equations (RDEs)

$$\dot{x} = f_\lambda(x, \xi_t) \tag{1}$$

$x \in M$ a smooth compact manifold.

Parameter $\lambda \in \mathbb{R}$ is varied.

ξ_t will be a realization of some noise process.

$f_\lambda(x, v)$ is a smooth vector field depending smoothly on $\lambda \in \mathbb{R}$ and $v \in \boldsymbol{\Delta}$.

ξ_t takes values in a closed disk $\boldsymbol{\Delta} \subset \mathbb{R}^n$.

H1 For each $x \in X$, $f_\lambda(x, \cdot)$ is a diffeomorphism with a convex range $f_\lambda(x, \boldsymbol{\Delta})$.

The Noise ξ_t

$\xi_t \in \mathcal{U} = \{\xi : \mathbb{R} \to \Delta, \xi \text{ measurable}\}$.

ξ_t is chosen "randomly" from $\mathcal{U}$.

The flow defined by the shift:

$$\theta^t : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \qquad \theta^t(\xi_s) = \xi_{s+t},$$

is then a continuous dynamical system with the weak topology on $\mathcal{U}$.

Existence, Uniqueness, Smooth Dependence

Since $\xi \in \mathcal{U}$ is measurable, and f is smooth and bounded, the differential equation (1) has unique, global solutions $\Phi^t_\lambda(x,\xi)$ (in the sense of Caratheodory), i.e.:

$$\Phi^t_\lambda(x,\xi) = x + \int_0^t f_\lambda(\Phi^t_\lambda(x,\xi), \xi_s)\, ds,$$

for any $\xi \in \mathcal{U}$ and all initial conditions x in X, and the solutions are absolutely continuous in t.

Solutions depend smoothly on λ.

Minimal Forward Invariant Sets

A set $F \subset X$ is *forward invariant* if

$$\Phi_\lambda^t(F, \mathcal{U}) \subset F \tag{2}$$

for all $t > 0$.

$\mathcal{F}$ – the collection of forward invariant sets.

There is a partial ordering on $\mathcal{F}$ by set inclusion.

We call $E \in \mathcal{F}$ a *minimal forward invariant* (MFI) set if it is minimal with respect to the partial ordering $\prec$.

MFI sets are forward orbits

Proposition 1 *Under assumption* **H1** *an MFI set for (1) is open and connected. The closures of distinct MFI sets are disjoint. If* x *is any point in an MFI set* E*, then* E *is equal to the forward orbit of* x*, i.e.*

$$E = O^{+}(x) \equiv \bigcup_{t>0} \Phi_{\lambda}^{t}(x,\mathcal{U}).$$

MFI sets support stationary measures

Under some additional assumptions on the noise, it can be shown that the closure of each MFI set coincides with the support of a stationary measure.

1-d MFI sets, $\dot{x} = f(x, \xi(t))$

Let: $f^{+}(x) = \max_{v \in \Delta} f(x,v)$, $f^{-}(x) = \min_{v \in \Delta} f(x,v)$

Proposition 2 *If* (a,b) *is an MFI set, then* $\forall x \in (a,b)$ *we have* $0 \in f(x, \Delta)$.

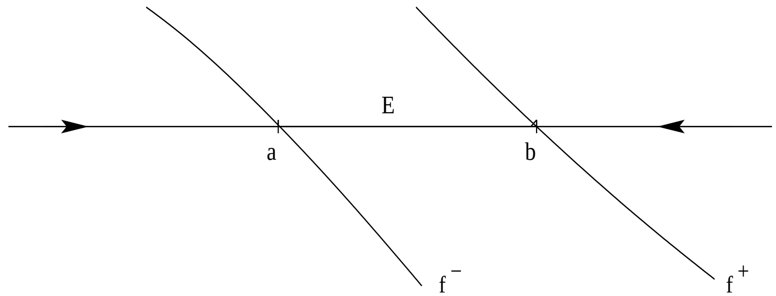
Proposition 3 *If* (a,b) *is an MFI set then*

$$f(a,v) \geq 0 \quad \textit{and} \quad f(b,v) \leq 0 \tag{3}$$

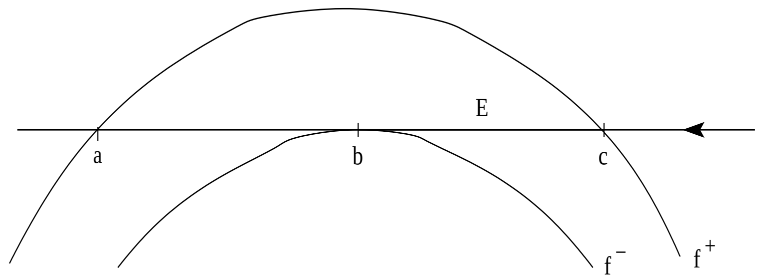
for all $v \in \Delta$. *Further* $f_{-}(a) = 0$, $f_{+}(b) = 0$, $f'_{-}(a) \leq 0$ *and* $f'_{+}(b) \leq 0$.

Examples of MFI sets in 1-d

(a)

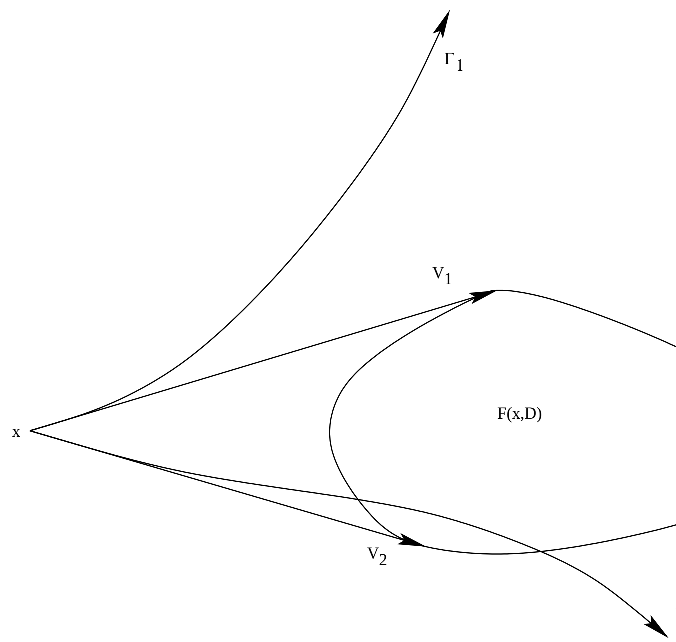


(b)



(a) A stable one dimensional MFI set. Both endpoints of $E = (a, b)$ are hyperbolic. (b) A random saddle-node in one dimension. $E = (b, c)$ is minimal forward invariant.

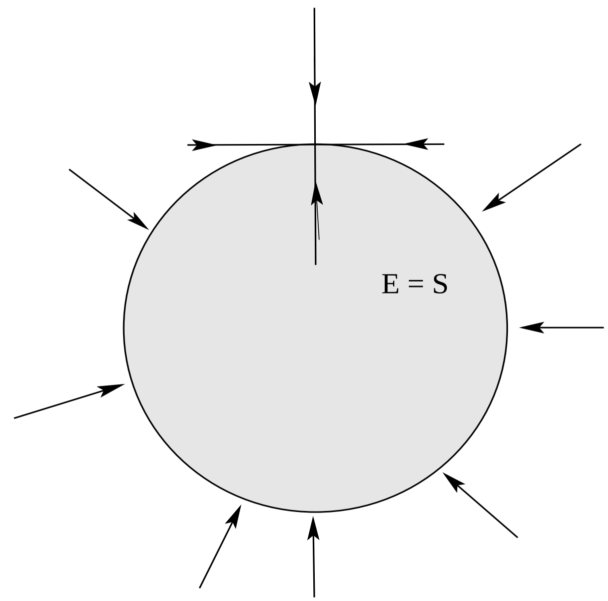
$\mathbb{R}^2$ - extreme vector fields and curves



Lemma 1 *MFI sets are bounded by extremal curves.*

Example. Perturbed linear stable node with repeated eigenvalues but distinct eigenvectors

$$\dot{x} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \epsilon u$$

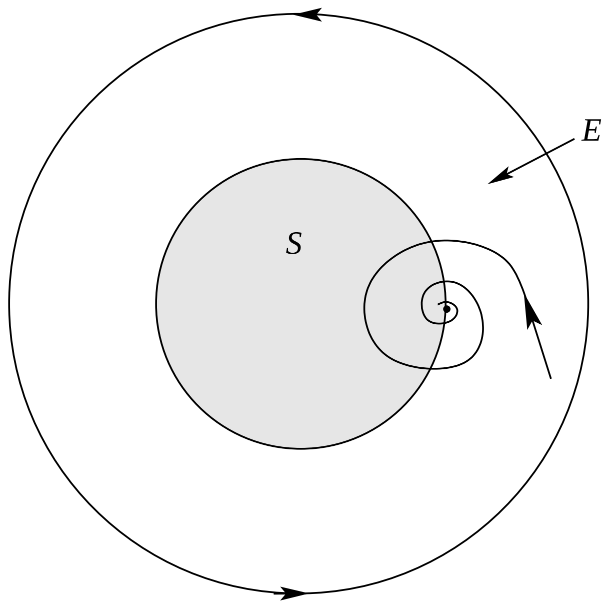


$S = \{x : 0 \in f(x, \Delta)\}$ – singular set.

In this case the MFI set E is S.

Example. Perturbed linear stable focus

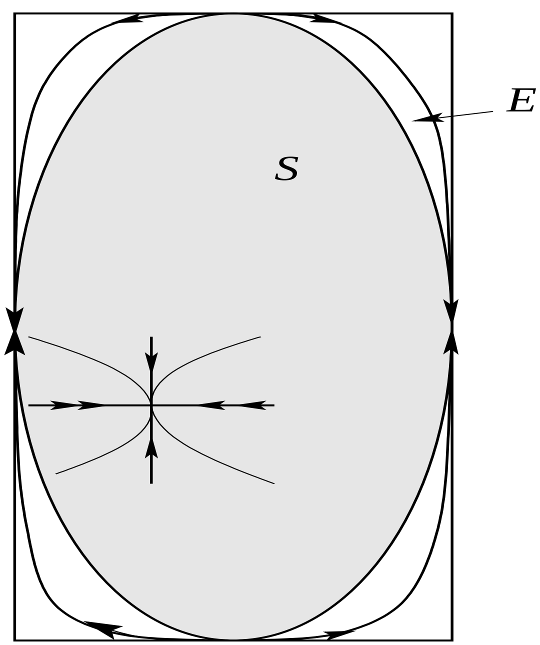
$$\dot{x} = -\begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix} x + \epsilon u$$



Example. Perturbed linear stable node with distinct eigenvalues

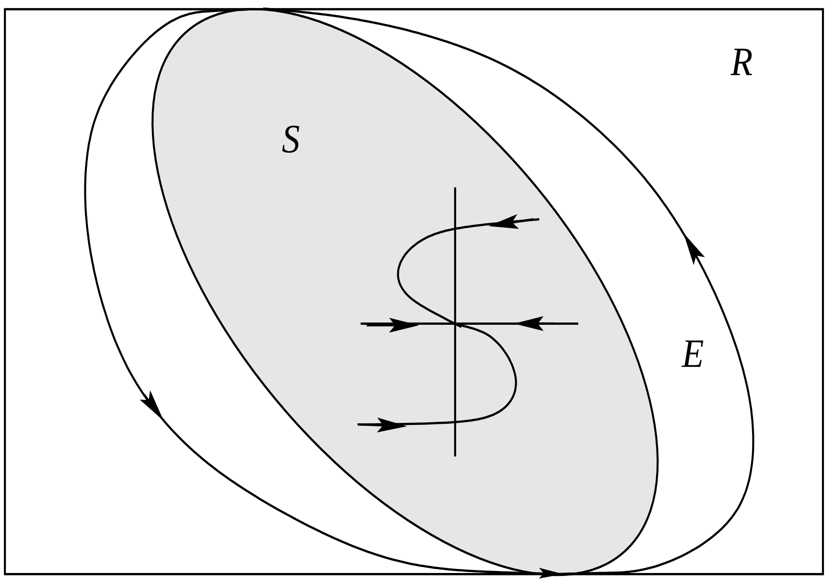
$$\dot{x} = -\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} x + \epsilon u$$

$0 < a < 1$



Example. Perturbed stable node with a single eigenvector

$$\dot{x} = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \epsilon u$$



Bifurcation of MFI sets

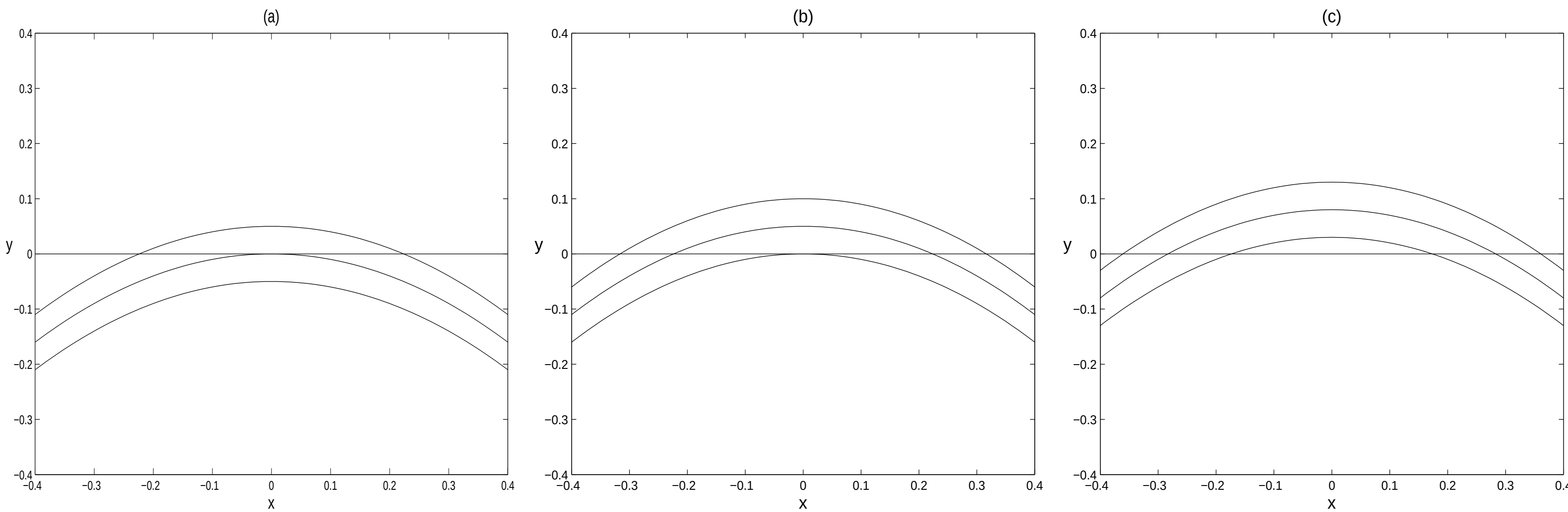
Definition 1 *A* bifurcation *of MFI sets is said to occur in a parameterized family of random differential equations if either:*

B1 *The number of MFI sets changes.*

B2 *An MFI set changes discontinuously with respect to the Hausdorff metric.*

Example. Saddle-node bifurcation in 1-d

$$\dot{x} = a - x^2 + \epsilon u$$



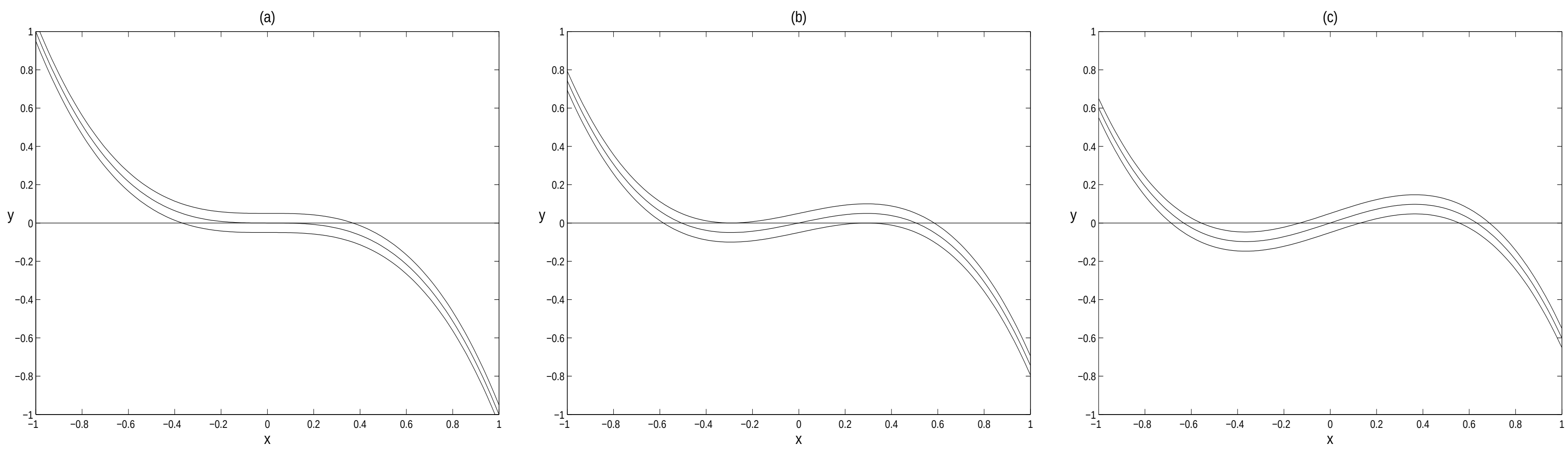
(a) The deterministic saddle-node bifurcation at $a = 0$. (b) The random bifurcation occurs at $a = \epsilon$. (c) For $a > \epsilon$ $\exists$ a trapping interval around the stable node.

Saddle-nodes in 1-d

Theorem 1 *The saddle-node is the only co-dimension one bifurcation in one dimensional RDE with bounded noise without symmetries.*

Pitchfork bifurcation

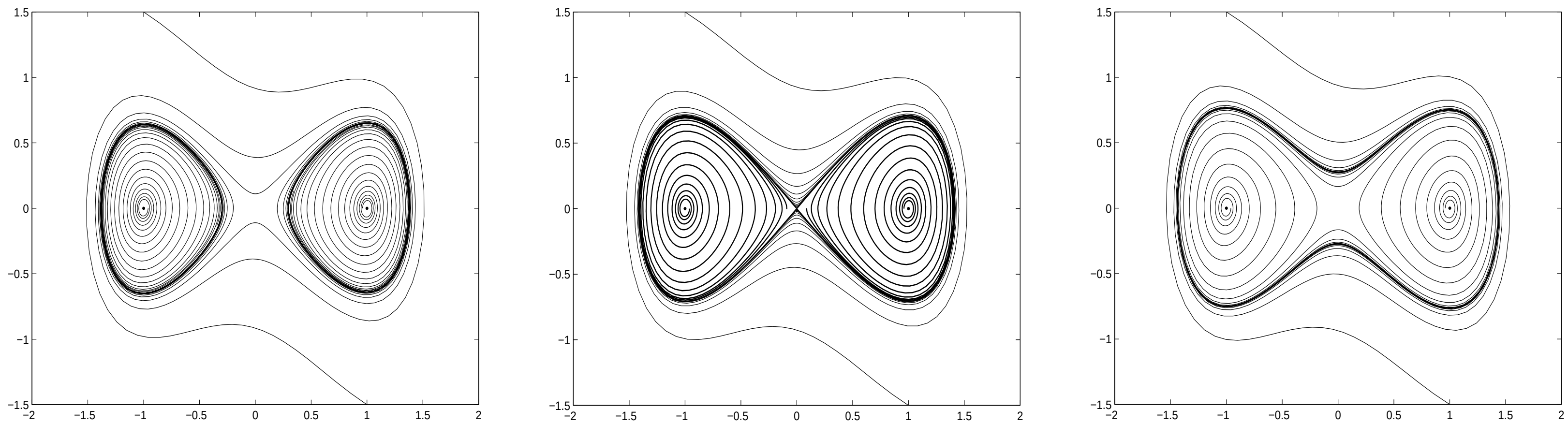
$$\dot{x} = ax - x^3 + \epsilon u$$



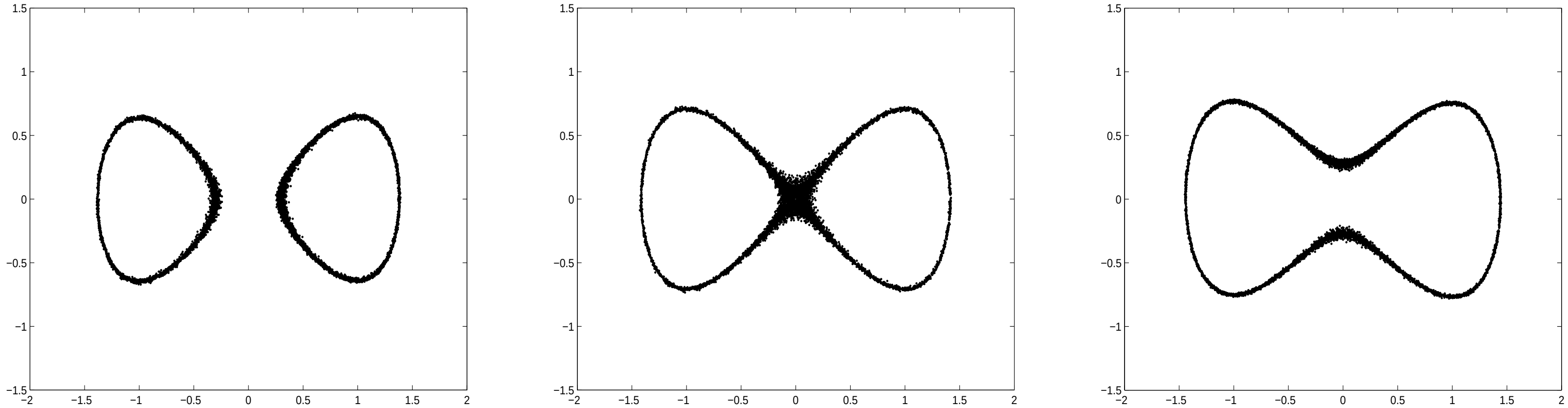
Stable homoclinic bifurcation in planar flows

$$X = y \qquad\qquad \dot{x} = X - \lambda Y$$
$$Y = x - x^3 + \frac{\delta}{2}(y^2 - x^2 + \frac{x^4}{2})y \qquad \dot{y} = \lambda X + Y$$

Deterministic homoclinic bifurcation:



Random homoclinic bifurcation



Phase portraits with $\delta = .6$ and added noise with level $\epsilon = .8$. In the first plot ($\lambda = -.01$) there is a pair of disjoint invariant densities. In the second plot ($\lambda = 0$), there is a single invariant density. In the third plot ($\lambda = .01$), the support of the invariant density has undergone a topological change.

Notion of Stability

Definition 2 *Denote by* R^∞ *the space of bounded noise vector fields* f *satisfying* **H1**. *Take as a topology on* R^∞ *the* C^∞ *topology on the vector fields* $f : X \times \boldsymbol{\Delta} \to TX$.

Definition 3 *An MFI set* E *for* f *is* stable *if there is a neighborhood* $U \supset E$ *such that if* $\tilde{f}$ *is sufficiently close to* f *in* R^∞ *then* $\tilde{f}$ *has exactly one MFI set* $\tilde{E} \subset U$ *and* $\tilde{E}$ *is close to* E *in the Hausdorff metric. We say that* $f \in R^\infty$ *is* stable *if all of its MFI sets* $\{E_i\}$ *are stable.*

Isolated MFI sets are stable.

Definition 4 *We say that an MFI set* E *for* (1) *is* isolated *if for any proper neighborhood* U *(*$\overline{E} \subset U$*) there is an open forward invariant set* $F \subset U$ *such that* $\overline{E} \subset F$*,* F *contains no other MFI set and* $\overline{\Phi_\lambda^t(F,\mathcal{U})} \subset F$ *for all* $t > 0$*.*

Theorem 2 *Isolated MFI sets are stable.*

Codimension one bifurcations in 2-d

Theorem 3 *There are three distinct codimension one bifurcations of bounded noise RDEs on compact surfaces. These are:*

- *Two sets of stationary points collide at a stationary point on the boundary ∂E which undergoes a saddle-node bifurcation.*
- *An MFI set E collides with a set of stationary points outside E at a saddle-point p.*
- *The Floquet multiplier of a non-isolated periodic cycle becomes one and then the cycle disappears.*