# Lecture 34

# Finite Difference Method – Nonlinear ODE

### Heat conduction with radiation

If we again consider the heat in a metal bar of length $L$, but this time consider the effect of radiation as well as conduction, then the steady state equation has the form

$$u_{xx} - d(u^4 - u_b^4) = -g(x), \tag{34.1}$$

where $u_b$ is the temperature of the background, $d$ incorporates a coefficient of radiation and $g(x)$ is the heat source.

If we again replace the continuous problem by its discrete approximation then we get

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - d(u_i^4 - u_b^4) = -g_i = -g(x_i). \tag{34.2}$$

This equation is nonlinear in the unknowns, thus we no longer have a system of linear equations to solve, but a system of nonlinear equations. One way to solve these equations would be by the multivariable Newton method. Instead, we introduce another interative method.

### Relaxation Method for Nonlinear Finite Differences

We can rewrite equation (34.2) as

$$u_{i+1} - 2u_i + u_{i-1} = h^2 d(u_i^4 - u_b^4) - h^2 g_i.$$

From this we can solve for $u_i$ in terms of the other quantities:

$$2u_i = u_{i+1} + u_{i-1} - h^2 d(u_i^4 - u_b^4) + h^2 g_i.$$

Next we add $u_i$ to both sides of the equation to obtain

$$3u_i = u_{i+1} + u_i + u_{i-1} - h^2 d(u_i^4 - u_b^4) + h^2 g_i,$$

and then divide by 3 to get

$$u_i = \frac{1}{3}\left(u_{i+1} + u_i + u_{i-1}\right) - \frac{h^2}{3}\left(d(u_i^4 - u_b^4) - g_i\right).$$

Now for the main idea. We will begin with an initial guess for the value of $u_i$ for each $i$, which we can represent as a vector $\mathbf{u}^0$. Then we will use the above equation to get better estimates, $\mathbf{u}^1$, $\mathbf{u}^2$, ..., and hope that they converge to the correct answer.

If we let

$$\mathbf{u}^j = (u_0^j, u_1^j, u_2^j, \ldots, u_{n-1}^j, u_n^j)$$

denote the $j$th approximation, then we can obtain that $j+1$st estimate from the formula

$$u_i^{j+1} = \frac{1}{3}\left(u_{i+1}^j + u_i^j + u_{i-1}^j\right) - \frac{h^2}{3}\left(d((u_i^j)^4 - u_b^4) - g_i\right).$$

Notice that $g_i$ and $u_b$ do not change. In the resulting equation, we have $u_i$ at each successive step depending on its previous value and the equation itself.

## Implementing the Relaxation Method

In the following program we solve the finite difference equations (34.2) with the boundary conditions $u(0) = 0$ and $u(L) = 0$. We let $L = 4$, $n = 4$, $d = 1$, and $g(x) = \sin(\pi x/4)$. Notice that the vector `u` always contains the current estimate of the values of $\mathbf{u}$.

```
% mynonlinheat (lacks comments)
% Purpose:
L = 4;              %
n = 100;              %
h = L/n;            %
hh = h^2/3;         %
u0 = 0;             %
uL = 0;             %
ub = .5;            %
ub4 = ub^4;         %
x = 0:h:L;          %
g = sin(pi*x/4);    %
u = zeros(1,n+1);   %
steps = 100;          %
u(1)=u0;            %
u(n+1)=uL;          %
plot(x,u)
hold on
for j = 1:steps
     %
     u(2:n) = (u(3:n+1)+u(2:n)+u(1:n-1))/3 + hh*(-u(2:n).^4+ub4+g(2:n));
     plot(x,u)
end
```

If you run this program with the given `n` and `steps` the result will not seem reasonable.

We can plot the initial guess by adding the command `plot(x,u)` right before the `for` loop. We can also plot successive iterations by moving the last `plot(x,u)` before the `end`. Now we can experiment and see if the iteration is converging. Try various values of *steps* and $n$ to produce a good plot. You will notice that this method converges quite slowly. In particular, as we increase $n$, we need to increase *steps* like $n^2$, i.e. if $n$ is large then steps needs to be *really* large.

## Exercises

34.1 Modify the script program `mynonlinheat` to plot the initial guess and all intermediate approximations. Add complete comments to the program. Print the program and a plot using $n = 12$ and `steps` large enough to see convergence.

34.2 Modify your improved `mynonlinheat` to `mynonlinheattwo` that has the boundary conditions

$$u(0) = 5 \quad \text{and} \quad u(L) = 10.$$

Fix the comments to reflect the new boundary conditions. Print the program and a plot using $n = 50$ and large enough `steps` to see convergence.