Lecture 33

ODE Boundary Value Problems and Finite Differences

Steady State Heat and Diffusion

If we consider the movement of heat in a long thin object (like a metal bar), it is known that the temperature, $u(x,t)$, at a location x and time t satisfies the partial differential equation

$$u_t - u_{xx} = g(x,t), \tag{33.1}$$

where $g(x,t)$ is the effect of any external heat source. The same equation also describes the diffusion of a chemical in a one-dimensional environment. For example the environment might be a canal, and then $g(x,t)$ would represent how a chemical is introduced.

Sometimes we are interested only in the steady state of the system, supposing $g(x,t) = g(x)$ and $u(x,t) = u(x)$. In this case

$$u_{xx} = -g(x).$$

This is a linear second-order ordinary differential equation. We could find its solution exactly if $g(x)$ is not too complicated. If the environment or object we consider has length L, then typically one would have conditions on each end of the object, such as $u(0) = 0$, $u(L) = 0$. Thus instead of an initial value problem, we have a **boundary value problem** or **BVP**.

Beam With Tension

Consider a simply supported beam with modulus of elasticity E, moment of inertia I, a uniform load w, and end tension T (see Figure 33.1). If $y(x)$ denotes the deflection at each point x in the beam, then $y(x)$ satisfies the differential equation

$$\frac{y''}{(1+(y')^2)^{3/2}} - \frac{T}{EI}y = \frac{wx(L-x)}{2EI}, \tag{33.2}$$

with boundary conditions $y(0) = y(L) = 0$. This equation is nonlinear and there is no hope to solve it exactly. If the deflection is small then $(y')^2$ is negligible compared to 1 and the equation approximately simplifies to

$$y'' - \frac{T}{EI}y = \frac{wx(L-x)}{2EI}. \tag{33.3}$$

This is a linear equation and we can find the exact solution. We can rewrite the equation as

$$y'' - \alpha y = \beta x(L-x), \tag{33.4}$$

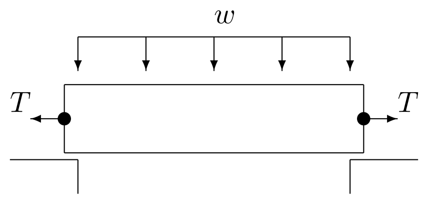

Figure 33.1: A simply supported beam with a uniform load w and end tension T.

where

$$\alpha = \frac{T}{EI} \quad \text{and} \quad \beta = \frac{w}{2EI}, \tag{33.5}$$

and then the exact solution is

$$y(x) = \frac{2\beta}{\alpha^2}\frac{e^{\sqrt{\alpha}L}}{e^{\sqrt{\alpha}L}+1}e^{-\sqrt{\alpha}x} + \frac{2\beta}{\alpha^2}\frac{1}{e^{\sqrt{\alpha}L}+1}e^{\sqrt{\alpha}x} + \frac{\beta}{\alpha}x^2 - \frac{\beta L}{\alpha}x + \frac{2\beta}{\alpha^2}. \tag{33.6}$$

Finite Difference Method – Linear ODE

A finite difference equation is an equation obtained from a differential equation by replacing the variables by their discrete versions and derivatives by difference formulas.

First we will consider equation (33.3). Suppose that the beam is a W12x22 structural steel I-beam. Then $L = 120$ in., $E = 29 \times 10^6$lb./in.^{2} and $I = 121$in.^{4}. Suppose that the beam is carrying a uniform load of $100,000$ lb. so that $w = 120,000/120 = 10,000$ and a tension of $T = 10,000$ lb.. We calculate from (33.5) $\alpha = 2.850 \times 10^{-6}$ and $\beta = 1.425 \times 10^{-6}$. Thus we have the following BVP:

$$y'' = 2.850 \times 10^{-6}y + 1.425 \times 10^{-6}x(120 - x), \qquad y(0) = y(120) = 0. \tag{33.7}$$

First subdivide the interval $[0, 120]$ into four equal subintervals. The nodes of this subdivion are $x_0 = 0$, $x_1 = 30$, $x_2 = 60$, $\ldots$, $x_4 = 120$. We will then let $y_0, y_1, \ldots, y_4$ denote the deflections at the nodes. From the boundary conditions we have immediately:

$$y_0 = y_4 = 0.$$

To determine the deflections at the interior points we will rely on the differential equation. Recall the central difference formula

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

In this case we have $h = (b - a)/n = (120 - 0)/4 = 30$. Replacing all the variables in the equation (33.4) by their discrete versions we get

$$y_{i+1} - 2y_i + y_{i-1} = h^2\alpha y_i + h^2\beta x_i(L - x_i).$$

Substituting in for α, β and h we obtain:

$$\begin{aligned} y_{i+1} - 2y_i + y_{i-1} &= 900 \times 2.850 \times 10^{-6}y_i + 900 \times 1.425 \times 10^{-6}x_i(120 - x_i) \\ &= 2.565 \times 10^{-3}y_i + 1.2825 \times 10^{-3}x_i(120 - x_i). \end{aligned}$$

This equation makes sense for $i = 1, 2, 3$. At $x_1 = 30$, the equation becomes:

$$\begin{aligned} y_2 - 2y_1 + y_0 &= 2.565 \times 10^{-3} y_1 + 1.2825 \times 10^{-3} \times 30(90) \\ \Leftrightarrow y_2 - 2.002565 y_1 &= 3.46275. \end{aligned} \tag{33.8}$$

Note that this equation is linear in the unknowns y_1 and y_2. At $x_2 = 60$ we have:

$$\begin{aligned} y_3 - 2y_2 + y_1 &= .002565 y_2 + 1.2825 \times 10^{-3} \times 60^2 \\ \Leftrightarrow y_3 - 2.002565 y_2 + y_1 &= 4.617. \end{aligned} \tag{33.9}$$

At $x_3 = 90$ we have (since $y_4 = 0$)

$$-2.002565 y_3 + y_2 = 3.46275. \tag{33.10}$$

Thus (y_1, y_2, y_3) is the solution of the linear system:

$$\left(\begin{array}{ccc|c} -2.002565 & 1 & 0 & 3.46275 \\ 1 & -2.002565 & 1 & 4.617 \\ 0 & 1 & -2.002565 & 3.46275 \end{array} \right).$$

We can easily find the solution of this system in MATLAB:

```
≫ A = [ -2.002565 1 0 ; 1 -2.002565 1 ; 0 1 -2.002565]
≫ b = [ 3.46275 4.617 3.46275 ]'
≫ y = A\b
```

To graph the solution, we need define the x values and add on the values at the endpoints:

```
≫ x = 0:30:120
≫ y = [0 ; y ; 0]
≫ plot(x,y,'d')
```

Adding a spline will result in an excellent graph.

The exact solution of this BVP is given in (33.6). That equation, with the parameter values for the W12x22 I-beam as in the example, is in the program `myexactbeam.m`. We can plot the true solution on the same graph:

```
≫ hold on
≫ myexactbeam
```

Thus our numerical solution is extremely good considering how few subintervals we used and how very large the deflection is.

An amusing exercise is to set $T = 0$ in the program `myexactbeam.m`; the program fails because the exact solution is no longer valid. Also try $T = .1$ for which you will observe loss of precision. On the other hand the finite difference method still works when we set $T = 0$.

Exercises

33.1 Derive the finite difference equations for the BVP (33.7) on the same domain ($[0, 120]$), but with eight subintervals and solve (using MATLAB) as in the example. Plot your result, together on the same plot with the exact solution (33.6) from the program `myexactbeam.m`.

33.2 By replacing y'' and y' with central differences, derive the finite difference equation for the boundary value problem

$$y'' + y' - y = x \quad \text{on } [0, 1] \quad \text{with} \quad y(0) = y(1) = 0$$

using 5 subintervals. Solve them and plot the solution using MATLAB.