Lecture 18

Iterative solution of linear systems*

Newton refinement

Conjugate gradient method
Review of Part II

Methods and Formulas

Basic Matrix Theory:

Identity matrix: \( AI = A, IA = A, \) and \( Iv = v \)
Inverse matrix: \( AA^{-1} = I \) and \( A^{-1}A = I \)
Norm of a matrix: \( \|A\| \equiv \max_{\|v\|=1} \|Av\| \)
A matrix may be singular or nonsingular. See Lecture 10.

Solving Process:

Gaussian Elimination produces LU decomposition
Row Pivoting
Back Substitution

Condition number:

\[
\text{cond}(A) \equiv \max \left( \frac{\|\delta x\|}{\|x\|} \right) = \max \left( \frac{\text{Relative error of output}}{\text{Relative error of inputs}} \right).
\]
A big condition number is bad; in engineering it usually results from poor design.

LU factorization:

\[
PA = LU.
\]
Solving steps:
Multiply by P: \( d = Pb \)
Forwardsolve: \( Ly = d \)
Backsolve: \( Ux = y \)

Eigenvalues and eigenvectors:

A nonzero vector \( v \) is an eigenvector and a number \( \lambda \) is its eigenvalue if

\[
Av = \lambda v.
\]
Characteristic equation: \( \det(A - \lambda I) = 0 \)
Equation of the eigenvector: \((A - \lambda I)v = 0\)
Residual for an approximate eigenvector-eigenvalue pair: \( r = \|Av - \lambda v\| \)

**Complex eigenvalues:**

Occur in conjugate pairs: \( \lambda_{1,2} = \alpha \pm i\beta \)
and eigenvectors must also come in conjugate pairs: \( w = u \pm iv \).

**Vibrational modes:**

Eigenvalues are frequencies squared. Eigenvectors represent modes.

**Power Method:**

- Repeatedly multiply \( x \) by \( A \) and divide by the element with the largest absolute value.
- The element of largest absolute value converges to largest absolute eigenvalue.
- The vector converges to the corresponding eigenvector.
- Convergence assured for a real symmetric matrix, but not for an arbitrary matrix, which may not have real eigenvalues at all.

**Inverse Power Method:**

- Apply power method to \( A^{-1} \).
- Use solving rather than the inverse.
- If \( \lambda \) is an eigenvalue of \( A \) then \( 1/\lambda \) is an eigenvalue for \( A^{-1} \).
- The eigenvectors for \( A \) and \( A^{-1} \) are the same.

**Symmetric and Positive definite:**

- Symmetric: \( A = A' \).
- If \( A \) is symmetric its eigenvalues are real.
- Positive definite: \( Ax \cdot x > 0 \).
- If \( A \) is positive definite, then its eigenvalues are positive.

**QR method:**

- Transform \( A \) into \( H \) the Hessian form of \( A \).
- Decompose \( H \) in \( QR \).
- Multiply \( Q \) and \( R \) together in reverse order to form a new \( H \).
- Repeat
- The diagonal of \( H \) will converge to the eigenvalues of \( A \).
Matlab

Matrix arithmetic:

\[ A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ -1 & -1 & 5 & 4 \\ 0 & 1 & -9 & 0 \end{bmatrix} \] .......................... Manually enter a matrix.
\[ u = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}' \]
\[ A*u \]
\[ B = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix} \] .......................... multiply \( B \) times \( A \).
\[ 2*A \] .......................... multiply a matrix by a scalar.
\[ A + A \] .......................... add matrices.
\[ A + 3 \] .......................... add 3 to every entry of a matrix.
\[ B.*B \] .......................... component-wise multiplication.
\[ B.^3 \] .......................... component-wise exponentiation.

Special matrices:

\[ I = \text{eye}(3) \] .......................... identity matrix
\[ D = \text{ones}(5,5) \]
\[ 0 = \text{zeros}(10,10) \]
\[ C = \text{rand}(5,5) \] .......................... random matrix with uniform distribution in \([0,1]\).
\[ C = \text{randn}(5,5) \] .......................... random matrix with normal distribution.
\[ \text{hilb}(6) \]
\[ \text{pascal}(5) \]

General matrix commands:

\[ \text{size}(C) \] .......................... gives the dimensions \((m \times n)\) of \( A \).
\[ \text{norm}(C) \] .......................... gives the norm of the matrix.
\[ \text{det}(C) \] .......................... the determinant of the matrix.
\[ \text{max}(C) \] .......................... the maximum of each row.
\[ \text{min}(C) \] .......................... the minimum in each row.
\[ \text{sum}(C) \] .......................... sums each row.
\[ \text{mean}(C) \] .......................... the average of each row.
\[ \text{diag}(C) \] .......................... just the diagonal elements.
\[ \text{inv}(C) \] .......................... inverse of the matrix.
\[ C' \] .......................... transpose of the matrix.

Matrix decompositions:

\[ [L \ U \ P] = \text{lu}(C) \]
\[ [Q \ R] = \text{qr}(C) \]
\[ H = \text{hess}(C) \] .......................... transform into a Hessian tri-diagonal matrix, which has the same eigenvalues as \( A \).
Part III
Functions and Data