Lecture 10

Some Facts About Linear Systems

Some inconvenient truths

In the last lecture we learned how to solve a linear system using Matlab. Input the following:

```
≫ A = ones(4,4)
≫ b = randn(4,1)
≫ x = A \ b
```

As you will find, there is no solution to the equation $A\mathbf{x} = \mathbf{b}$. This unfortunate circumstance is mostly the fault of the matrix, A, which is too simple, its columns (and rows) are all the same. Now try

```
≫ b = ones(4,1)
≫ x = [ 1 0 0 0]'
≫ A*x
```

So the system $A\mathbf{x} = \mathbf{b}$ does have a solution. Still unfortunately, that is not the only solution. Try

```
≫ x = [ 0 1 0 0]'
≫ A*x
```

We see that this x is also a solution. Next try

```
≫ x = [ -4  5  2.27 -2.27]'
≫ A*x
```

This x is a solution! It is not hard to see that there are endless possibilities for solutions of this equation.

Basic theory

The most basic theoretical fact about linear systems is

Theorem 1 *A linear system $A\mathbf{x} = \mathbf{b}$ may have 0, 1, or infinitely many solutions.*

In most (but not all!) engineering applications we would want to have exactly one solution. The following two theorems tell us exactly when we can and cannot expect this.

Theorem 2 *Suppose A is a square $(n \times n)$ matrix. The following are all equivalent:*

1. *The equation* $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for any* $\mathbf{b}$.
2. $det(A) \neq 0$.
3. *A has an inverse.*
4. *The only solution of* $A\mathbf{x} = \mathbf{0}$ *is* $\mathbf{x} = \mathbf{0}$.
5. *The columns of A are linearly independent (as vectors).*
6. *The rows of A are linearly independent.*

If A has these properties then it is called **non-singular**.

On the other hand, a matrix that does not have these properties is called **singular**.

Theorem 3 *Suppose A is a square matrix. The following are all equivalent:*

1. *The equation* $A\mathbf{x} = \mathbf{b}$ *has 0 or* ∞ *many solutions depending on* $\mathbf{b}$.
2. $\det(A) = 0$.
3. *A does not have an inverse.*
4. *The equation* $A\mathbf{x} = \mathbf{0}$ *has solutions other than* $\mathbf{x} = \mathbf{0}$.
5. *The columns of A are linearly dependent as vectors.*
6. *The rows of A are linearly dependent.*

To see how the two theorems work, define two matrices (type in `A1` then scroll up and modify to make `A2`) :

$$\texttt{A1} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad \texttt{A2} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix},$$

and two vectors:

$$\texttt{b1} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}, \qquad \texttt{b2} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.$$

First calculate the determinants of the matrices:

```
>> det(A1)
>> det(A2)
```

Then attempt to find the inverses:

```
>> inv(A1)
>> inv(A2)
```

Which matrix is singular and which is non-singular? Finally, attempt to solve all the possible equations $A\mathbf{x} = \mathbf{b}$:

```
>> x = A1 \ b1
>> x = A1 \ b2
>> x = A2 \ b1
>> x = A2 \ b2
```

As you can see, equations involving the non-singular matrix have one and only one solution, but equation involving a singular matrix are more complicated.

The Residual

Recall that the residual for an approximate solution x of an equation $f(x) = 0$ is defined as $r = \|f(x)\|$. It is a measure of how close the equation is to being satisfied. For a linear system of equations we define the residual vector of an approximate solution $\mathbf{x}$ by

$$\mathbf{r} = A\mathbf{x} - \mathbf{b}.$$

If the solution vector $\mathbf{x}$ were exactly correct, then $\mathbf{r}$ would be exactly the zero vector. The size (norm) of $\mathbf{r}$ is an indication of how close we have come to solving $A\mathbf{x} = \mathbf{b}$. We will refer to this number as the **scalar residual** or just the **residual** of the approximation solution:

$$r = \|A\mathbf{x} - \mathbf{b}\|. \tag{10.1}$$

Exercises

10.1 By hand, find all the solutions (if any) of the following linear system using the augmented matrix and Gaussian elimination (following exactly the algorithm in the notes):

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4, \\ 4x_1 + 5x_2 + 6x_3 &= 10, \\ 7x_1 + 8x_2 + 9x_3 &= 14\,. \end{aligned}$$

Try solving this system in MATLAB using the command `x = A \ b`. What happens? Turn in your hand work.

10.2 (a) Write a well-commented MATLAB **function** program `mysolvecheck` with input a number `n` that makes a random $n \times n$ matrix A and a random vector $\mathbf{b}$, solves the linear system $A\mathbf{x} = \mathbf{b}$, calculates the scalar residual $r = \|A\mathbf{x} - \mathbf{b}\|$, and outputs that number as r.

(b) Write a well-commented MATLAB **script** program that calls `mysolvecheck` 10 times each for $n = 5$, 10, 20, 40, 80, and 160, then records and averages the results and makes a log-log plot of the average r vs. n. Once your program is running correctly, increase the maximum n (by factors of 2) until the program stops running within 5 minutes.

Turn in the plot and the two programs. (Do not print any large random matrices.)