

Lecture 10

Some Facts About Linear Systems

Some inconvenient truths

In the last lecture we learned how to solve a linear system using Matlab. Input the following:

```
>> A = ones(4,4)
>> b = randn(4,1)
>> x = A \ b
```

As you will find, there is no solution to the equation $A\mathbf{x} = \mathbf{b}$. This unfortunate circumstance is mostly the fault of the matrix, A , which is too simple, its columns (and rows) are all the same. Now try

```
>> b = ones(4,1)
>> x = [ 1 0 0 0] '
>> A*x
```

So the system $A\mathbf{x} = \mathbf{b}$ does have a solution. Still unfortunately, that is not the only solution. Try

```
>> x = [ 0 1 0 0] '
>> A*x
```

We see that this x is also a solution. Next try

```
>> x = [ -4 5 2.27 -2.27] '
>> A*x
```

This x is a solution! It is not hard to see that there are endless possibilities for solutions of this equation.

Basic theory

The most basic theoretical fact about linear systems is

Theorem 1 *A linear system $A\mathbf{x} = \mathbf{b}$ may have 0, 1, or infinitely many solutions.*

In most (but not all!) engineering applications we would want to have exactly one solution. The following two theorems tell us exactly when we can and cannot expect this.

Theorem 2 *Suppose A is a square ($n \times n$) matrix. The following are all equivalent:*

1. The equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any \mathbf{b} .
2. $\det(A) \neq 0$.
3. A has an inverse.
4. The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
5. The columns of A are linearly independent (as vectors).
6. The rows of A are linearly independent.

If A has these properties then it is called **non-singular**.

On the other hand, a matrix that does not have these properties is called **singular**.

Theorem 3 Suppose A is a square matrix. The following are all equivalent:

1. The equation $A\mathbf{x} = \mathbf{b}$ has 0 or ∞ many solutions depending on \mathbf{b} .
2. $\det(A) = 0$.
3. A does not have an inverse.
4. The equation $A\mathbf{x} = \mathbf{0}$ has solutions other than $\mathbf{x} = \mathbf{0}$.
5. The columns of A are linearly dependent as vectors.
6. The rows of A are linearly dependent.

To see how the two theorems work, define two matrices (type in **A1** then scroll up and modify to make **A2**) :

$$A1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix},$$

and two vectors:

$$b1 = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}, \quad b2 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.$$

First calculate the determinants of the matrices:

» `det(A1)`
 » `det(A2)`

Then attempt to find the inverses:

» `inv(A1)`
 » `inv(A2)`

Which matrix is singular and which is non-singular? Finally, attempt to solve all the possible equations $A\mathbf{x} = \mathbf{b}$:

```

>> x = A1 \ b1
>> x = A1 \ b2
>> x = A2 \ b1
>> x = A2 \ b2

```

As you can see, equations involving the non-singular matrix have one and only one solution, but equation involving a singular matrix are more complicated.

The Residual

Recall that the residual for an approximate solution x of an equation $f(x) = 0$ is defined as $r = \|f(x)\|$. It is a measure of how close the equation is to being satisfied. For a linear system of equations we define the residual vector of an approximate solution \mathbf{x} by

$$\mathbf{r} = A\mathbf{x} - \mathbf{b}.$$

If the solution vector \mathbf{x} were exactly correct, then \mathbf{r} would be exactly the zero vector. The size (norm) of \mathbf{r} is an indication of how close we have come to solving $A\mathbf{x} = \mathbf{b}$. We will refer to this number as the **scalar residual** or just the **residual** of the approximation solution:

$$r = \|A\mathbf{x} - \mathbf{b}\|. \quad (10.1)$$

Exercises

- 10.1 By hand, find all the solutions (if any) of the following linear system using the augmented matrix and Gaussian elimination (following exactly the algorithm in the notes):

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4, \\ 4x_1 + 5x_2 + 6x_3 &= 10, \\ 7x_1 + 8x_2 + 9x_3 &= 14. \end{aligned}$$

Try solving this system in MATLAB using the command $\mathbf{x} = A \setminus \mathbf{b}$. What happens? Turn in your hand work.

- 10.2 (a) Write a well-commented MATLAB **function** program `mysolvecheck` with input a number n that makes a random $n \times n$ matrix A and a random vector \mathbf{b} , solves the linear system $A\mathbf{x} = \mathbf{b}$, calculates the scalar residual $r = \|A\mathbf{x} - \mathbf{b}\|$, and outputs that number as r .
- (b) Write a well-commented MATLAB **script** program that calls `mysolvecheck` 10 times each for $n = 5, 10, 20, 40, 80$, and 160, then records and averages the results and makes a log-log plot of the average r vs. n . Once your program is running correctly, increase the maximum n (by factors of 2) until the program stops running within 5 minutes.

Turn in the plot and the two programs. (Do not print any large random matrices.)