Logical Connectives

Mathematics has its own language. As with any language, effective communication depends on logically connecting components. Even the simplest "real" mathematical problems require at least a small amount of reasoning, so it is very important that you develop a feeling for *formal (mathematical) logic*.

Consider, for example, the two sentences "There are 10 people waiting for the bus" and "The bus is late." What, if anything, is the logical connection between these two sentences? Does one *logically imply* the other? Similarly, the two mathematical statements "$r^2 + r - 2 = 0$" and "$r = 1$ or $r = -2$" need to be connected, otherwise they are merely two random statements that convey no useful information. Warning: when mathematicians talk about implication, it means that one thing **must** be true as a consequence of another; not that it can be true, or might be true sometimes.

Words and symbols that tie statements together logically are called *logical connectives*. They allow you to communicate the reasoning that has led you to your conclusion. Possibly the most important of these is *implication* — the idea that the next statement is a logical consequence of the previous one. This concept can be conveyed by the use of words such as: therefore, hence, and so, thus, since, if ... then ..., this implies, etc. In the middle of mathematical calculations, we can represent these by the implication symbol ($\Rightarrow$). For example

$$x + 7y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{\frac{3-x}{7}}\,; \tag{1}$$

$$x \in (0,\infty) \quad \Rightarrow \quad \cos(x) \in [-1,1]. \tag{2}$$

Converse

Note that "statement A $\Rightarrow$ statement B" does **not** necessarily mean that the *logical converse* — "statement B $\Rightarrow$ statement A" — is also true. Logical implication is a matter of cause and effect; the logical converse is simply the reverse cause-effect situation (which may not be true). Consider the following, everyday example:

- "I am running to class because I am late."

It should be clear that the logical converse of these is not true:

- "I am late to class because I am running."

For examples (1) and (2) above:

$$x + 7y^2 = 3 \quad \Leftarrow \quad y = \pm\sqrt{\frac{3-x}{7}}\,;$$

$$x \in (0,\infty) \quad \not\Leftarrow \quad \cos(x) \in [-1,1]. \quad \boxed{\text{Since } x < 0 \Rightarrow \cos(x) \in [-1,1] \text{ also.}}$$

Contrapositive

We have seen that simply reversing a logical statement can lead to problems. There is a way, however, to invert an implication so that the inverted statement is also true. This is known as the *contrapositive*. Consider the statement "$A \Rightarrow B$"; this means "if A is true, then B is true". The contrapositive is "if B is **false**, then A must be false".

Here is an example of how these concepts fit together:

- If X is a cat, it is an animal. $\boxed{\text{Accept as true}}$
- If X is an animal, it is a cat. $\boxed{\text{FALSE (converse)}}$
- If X is not an animal, it is not a cat. $\boxed{\text{TRUE (contrapositive)}}$
- If X is not a cat, it is not an animal.
  $\boxed{\text{FALSE. This is the contrapositive of the second statement, which is false.}}$

Equivalence

When the implication works both ways, we say that the two statements are *equivalent* and we may use the equivalence symbol ($\Leftrightarrow$); in words we may say "A is equivalent to B" or "A if and only if B". If two statements are equivalent, we may use any of the implication symbols ($\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$). Which connective we use depends on what we are trying to show. In (1) above, if we are trying to obtain a formula for y, we would probably just use "$\Rightarrow$", even though the stronger statement ("$\Leftrightarrow$") is also true. If, however, we wanted to show that two statements about x and y were equivalent, we would write $x + 7y^2 = 3 \Leftrightarrow y = \pm\sqrt{\frac{3-x}{7}}$.

Careful logic is the heart and soul of mathematics; learn to reason with watertight arguments and use logical connectives to explain your reasoning processes.

Example:

What is the greatest amount of water that a right-cylindrical water tank can hold if there is $100m^2$ of material from which to construct it?

Good: Let the height of the tank be h and the radius be r. Then the volume of the tank is $V(h,r) = \pi h r^2$ and the surface area is $S(h,r) = 2\pi r^2 + 2\pi r h = 2\pi r(r+h)$. Since we require $V > 0$, we can assume that $r, h > 0$. If we have $100m^2$ of material, then $S = 100$, and so

$$\begin{aligned} S &= 2\pi r(r+h) = 100 \\ &\Rightarrow r + h = \frac{100}{2\pi r} \\ &\Rightarrow h = \frac{50}{\pi r} - r \\ &\Rightarrow V = \pi h r^2 = 50r - \pi r^3. \end{aligned}$$

To find the maximum volume, we look for r such that $\frac{dV}{dr} = 0$. Differentiating with respect to r gives $\frac{dV}{dr} = 50 - 3\pi r^2$. Hence,

$$\begin{aligned} \frac{dV}{dr} = 0 &\Leftrightarrow 50 - 3\pi r^2 = 0 \\ &\Leftrightarrow r^2 = \frac{50}{3\pi} \\ &\Leftrightarrow r = \pm\sqrt{\frac{50}{3\pi}}. \end{aligned}$$

Since $r > 0$, we can use the second derivative test and find $\frac{d^2V}{dr^2} = -6\pi r < 0$. This implies that $r = \sqrt{\frac{50}{3\pi}} \approx 2.3033$ maximizes V. From above, the optimal volume is

$$\begin{aligned} V &= 50r - \pi r^3 \\ &= \left(50 - \pi\left(\frac{50}{3\pi}\right)\right)\sqrt{\frac{50}{3\pi}} \\ &= \left(\frac{100}{3}\right)\sqrt{\frac{50}{3\pi}} \approx 76.7765. \end{aligned}$$

Thus we have found that the greatest amount of water such a tank can hold is (approximately) $76.7765m^3$.