1 Week of April 15th, 2009

Derive the $L^2 \rightarrow L^2_\nabla$ norm bound on integral operator $\kappa$.

Let $\kappa$ denote an integral operator such that:

$$\kappa q(r) = \int K(r, r') q(r') dr',$$

where the kernel is expressed in an orthonormal basis as:

$$K(r, r') = \sum_i \sum_{i'} c_{i, i'} \phi_i(r) \phi_{i'}(r')$$

Now our goal is to find the smallest number $||\kappa||_{2, \nabla_2}$ (bound) such that:

$$||\nabla \kappa q(r)||_2 \leq ||\kappa||_{2, \nabla_2} \cdot ||q||_2.$$

Derive:

$$||\nabla \kappa q(r)||_2^2 = \int (\nabla \kappa q(r))^2 dr$$

$$= \int (\nabla r \int K(r, r') q(r') dr')^2 dr$$

$$= \int (\nabla r \int (\sum_i \sum_{i'} C_{i, i'} \phi_i(r) \phi_{i'}(r')) q(r') dr')^2 dr$$

$$= \int (\sum_i \nabla r \phi_i(r) \sum_{i'} \int C_{i, i'} \phi_{i'}(r') q(r') dr')^2 dr$$

$$\leq \int (\sum_i \nabla r \phi_i(r) \sum_{i'} \|C_{i, i'} \phi_{i'}(r')\|_2 \cdot ||q(r')||_2)^2 dr$$

1
\[
\int (\sum_i \nabla_r \phi_i(r) \sum_j |C_{ij} \phi_i'(r')|^2 ||q(r')||_2^2 dr \\
= ||q(r')||_2^2 \int (\sum_i \nabla_r \phi_i(r) \sum_j |C_{ij} \phi_i'(r')|^2 ||q(r')||_2^2 dr
\]

Therefore, \( ||\nabla \kappa q(r)||_2 \leq (\int (\sum_i \nabla_r \phi_i(r) \sum_j |C_{ij} \phi_i'(r')|^2 ||q(r')||_2^2 dr)^{\frac{1}{2}} \cdot ||q||_2 \).

2 Week of April 22\(^{th}\), 2009

Last week’s work is wrong at (6), because \( \nabla_r \phi_i(r) \) may be negative.

Now derive:

\[
||\nabla \kappa q(r)||_2 = (\int (\nabla_r (\kappa q(r)))^2 dr)^{\frac{1}{2}}
\]

\[
= (\int (\int K(r, r') q(r') dr')^2 dr)^{\frac{1}{2}}
\]

\[
= (\int (\int K(r, r') q(r') dr') \cdot (\int K(r, r''') q(r''') dr') dr)^{\frac{1}{2}}
\]

\[
= (\int q(r') dr' \int (\int \nabla_r K(r, r') \cdot \nabla_r K(r, r''') dr') q(r''') dr')^{\frac{1}{2}}
\]

Defining the self-adjoint integral operator \( \nabla \kappa \ast \nabla \kappa \) by:

\[
(\nabla K \ast \nabla K)(r', r''') = \int \nabla K(r, r') \cdot \nabla K(r, r''') dr,
\]

Using Holder inequality with \( p = q = 2 \) we have:

\[
(\int q(r') dr' \int (\nabla K \ast \nabla K) q(r''') dr')^{\frac{1}{2}} = \langle q, (\nabla K \ast \nabla K) q \rangle^{\frac{1}{2}} \leq ||q||_2^{\frac{1}{2}} ||\nabla K \ast \nabla K||_2^{\frac{1}{2}} \leq ||\nabla K \ast \nabla K||_2^{\frac{1}{2}} ||q||_2
\]

Since \( \nabla \kappa \ast \nabla \kappa \) is self-adjoint, its \( L^2 \to L^2 \) norm is its spectral radius \( \rho(\nabla \kappa \ast \nabla \kappa) \), which is the maximum value among the absolute value of the eigenvalues of \( \nabla \kappa \ast \nabla \kappa \), that is:

\[
||\kappa||_{2, \nabla_2} = ||\nabla \kappa \ast \nabla \kappa||_2^{\frac{1}{2}} = \sqrt{\rho(\nabla \kappa \ast \nabla \kappa)}
\]

In terms of its coefficients, we have:

\[
(\nabla \kappa \ast \nabla K)(r', r''')
\]
duality, we have:

\[ \kappa = \int \left( \sum_i \sum_{j'} c_{i,j'} \phi_j(r) \nabla r \phi_i(r) \right) \left( \sum_j \sum_{j''} c_{j,j''} \phi_j(r) \nabla r \phi_j(r) \right) dr \quad (17) \]

\[ = \sum_i \sum_{j'} \phi_{i,j'}(r) \phi_j(r) \left( \sum_j \sum_{j''} c_{j,j''} c_{i,i'} \int \nabla \phi_i(r) \cdot \nabla \phi_j(r) dr \right) \quad (18) \]

3 Week of April 29th, 2009

Derive the \( L^2 \to L^2 \) norm bound on error operator.

We decompose the space into disjoint blocks indexed by \( l \).

Let \( \kappa \) be the error operator, and

\[ \kappa q(r) = \int K(r, r')q(r') dr', \]

where \( K(r, r') = \sum_i \sum_{i'} \chi^l(r) K(r, r') \chi^{i'}(r') = \sum_i \sum_{i'} \chi^l(r) K^{i-l'}(r, r') \chi^{i'}(r') \)

Our goal is to find \( ||\kappa||_{2, \nabla_2} \) such that \( ||\nabla \kappa q||_2 \leq ||\kappa||_{2, \nabla_2}||q||_2 \). Using \( (L^2, L^2) \) duality, we have:

\[ ||\kappa||_{2, \nabla_2} \]
\[ = \sup_{||p||=1, ||q||=1} \langle p, \nabla r \kappa q \rangle \]
\[ = \sup_{||p||=1, ||q||=1} \int p(r) \nabla r \int K(r, r')q(r') dr' dr \]
\[ = \int p(r) \int \nabla r (r, r')q(r') dr' dr \]
\[ = \sup_{||p||=1, ||q||=1} \int p(r) \int \sum_i \sum_{i'} \chi^l(r) \nabla r K^{i-l'}(r, r') \chi^{i'}(r')q(r') dr' dr \]
\[ = \sup_{||p||=1, ||q||=1} \sum_i \sum_{i'} \int \chi^l(r)p(r) \nabla r K^{i-l'}(r, r') \chi^{i'}(r')q(r') dr' dr \]
\[ = \sup_{||p||=1, ||q||=1} \sum_i \sum_{i'} \langle p, \nabla r K^{i-l'}(r, r') \chi^{i'}(r')q(r') dr' \rangle \]

Let \( x = \chi^l(r)p(r) \), \( y = \chi^{i'}(r')q(r') \), and \( A = \int \nabla r K^{i-l'}(r, r')y dr' \).

Then,

\[ | \langle x, Ay \rangle | \leq ||x||_2 ||A||_{2, \nabla_2} ||y||_2. \]

That is,

\[ | \langle x, Ay \rangle | \leq ||x||_2 ||A||_{2, \nabla_2} ||y||_2. \]
Now, we continue and get:

\[
\leq \sup_{|p|_2=|q|_2=1} \sum_l \sum_{p'} ||X^l p||_2 ||\nabla \kappa^{l-l'}||_2 ||X^{l'} q||_2 \\
= \sup_{|p|_2=|q|_2=1} \sum_l \sum_{m} ||X^l p||_2 ||\nabla \kappa^m||_2 ||X^{l+m} q||_2 \\
= \sum_m ||\nabla \kappa^m||_2 \sup_{|p|_2=|q|_2=1} \sum_l ||X^l p||_2 ||X^{l+m} q||_2 \\
= \sum_m ||\nabla \kappa^m||_2 \sup_{|p|_2=|q|_2=1} < ||X^l p||_2, ||X^{l+m} q||_2 > \geq^2 \\
= \sum_m ||\nabla \kappa^m||_2 ||X^l p||_2 ||X^{l+m} q||_2 = \sum_m ||\nabla \kappa^m||_2.
\]

For every single \(||\nabla \kappa||_2\), it’s defined as its spectral radius \(\rho(\nabla \kappa \ast \nabla \kappa)\), which is the maximum value among the absolute value of its eigenvalues, that is:

\[
||\nabla \kappa||_2 = ||\nabla \kappa \ast \nabla \kappa||^\frac{1}{2} = \sqrt{\rho(\nabla \kappa \ast \nabla \kappa)}
\]

## 4 Week of May 06th, 2009

We go back to last quarter’s work: try to figure out the best constant \(K\) such that \(\|\phi\theta\|_2 \leq K \cdot f(\|\phi\|_{H^1}, \|\theta\|_{H^1}, \|\phi\|_2, \|\theta\|_2)\). What we did last quarter was to derive as follows:

\[
\|\phi\theta\|^4 \leq \|\phi^2\|^2 \cdot \|\theta^2\|^2 \\
= \|\phi\|^4 \cdot \|\theta\|^4 \\
\leq \|\phi\|^2 \cdot \|\phi\|^2 \cdot \|\theta\|^2 \cdot \|\theta\|^2
\]

A book titled ”Aspects of Sobolev-Type Inequalities” by Laurent Saloff-Coste (Page 20) gives the best constant in the Sobolev Inequality \(\|f\|_q \leq C(n, p) \cdot \|\nabla f\|_p\), when \(f \in C^\infty_0(R^n)\), \(1 \leq p < n\) and \(q = np/(n-p)\):

\[
C(n, p) = \frac{p-1}{n-p} \left( \frac{n-p}{n(p-1)} \right)^\frac{1}{2} \Gamma(n+1) \frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}^{\frac{1}{2}}
\]

For Gamma Function:

When \(n \in Z^+\),

\[
\Gamma(n+1) = n!
\]

When \(z\) is a complex number with positive real part,

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt
\]
On the other hand, \( \omega_{n-1} = n \Omega_n \), where \( \Omega_n \) is the volume of the unit ball. For our problem, we need to compute for \( C(3, 2) \).

\[
\Gamma(4) = 3! = 6 \quad (25)
\]

\[
\Gamma(1.5) = \int_0^\infty t^{0.5} e^{-t} dt = \frac{1}{2\pi^{1/2}} \quad (26)
\]

\[
\Gamma(2.5) = \int_0^\infty t^{1.5} e^{-t} dt = \frac{3}{4\pi^{1/2}} \quad (27)
\]

\[
\omega_2 = 3\pi^{1/2} \quad (28)
\]

As to \( \omega_{n-1} \) when \( n=3 \), \( \Omega_3 = \frac{4}{3}\pi \) so that \( \omega_{3-1} = 3 \cdot \frac{4}{3}\pi = 4\pi. \) Hence,

\[
C(3, 2) = \frac{2 - 1}{3 - 2} \cdot \frac{3 - 2}{3(2 - 1)} \cdot \left( \frac{\Gamma(3 + 1)}{\Gamma(3/2)\Gamma(3 + 1 - 3/2)\omega_{3-1}} \right)^{1/2} = (\frac{1}{3})^{1/2} \cdot \left( \frac{6}{2\pi \cdot \frac{3}{2} \pi \cdot 4\pi} \right)^{1/2} = 0.616216 \quad (29)
\]

Therefore,

\[
\| \phi \theta \|_2 \leq C(3, 2)^{3/2} \cdot \| \phi \|_2 \cdot \| \nabla \phi \|_2 \cdot \| \theta \|_2 \cdot \| \nabla \theta \|_2 \approx 0.483726 \| \phi \|_2 \cdot \| \nabla \phi \|_2 \cdot \| \theta \|_2 \cdot \| \nabla \theta \|_2
\]

Our questions are:
1) The book states that equation (4) gives the best constant for the sobolev inequality \( \| f \|_q \leq C(n, p) \cdot \| \nabla f \|_p \) when \( f \in C_0^\infty (\mathbb{R}^n) \), \( 1 \leq p < n \) and \( q = np/(n - p) \). We want to make sure that it’s the best.
2) When we derived the constant K such that \( \| \phi \theta \|_2 \leq K \cdot f(\| \phi \|_{H^1}, \| \theta \|_{H^1}, \| \nabla \phi \|_2, \| \nabla \theta \|_2) \), we used \( L^6 \) as a bridge (see equation (3)) and then we went back to \( L^2 \) by equation(4). We are wondering if there’s any way that we can skip this tour and get a better constant K.

5 Week of May 13th, 2009

I made an appointment with Professor Guli talking about the sharp bound \( C \) such that \( \| \phi \theta \|_2 \leq C \cdot f(\| \phi \|_2, \| \theta \|_2, \| \nabla \phi \|_2, \| \nabla \theta \|_2). \) His answer was that the bound \( C \) depends on what kind of function \( f \) derived and he did not know the exact sharp bound for a certain function \( f \). Another question is the assumptions of interchange unbounded integral with gradient that is used to derive the bound on integral operator. Fan asked Professor Achi, however, the conditions he gave are just too strong.
6 Week of May 20\textsuperscript{th}, 2009

Fan and I made a report on using Sobolev Spaces. We listed whatever we have done so far with Dr. Martin. Dr. Martin gave us some suggestions to fix our report and encouraged us to give a final presentation next Wednesday during the meeting.