1 Week of September 4th, 2007

1.1 LaTex

I tried to write something with LaTex and had some difficulties with its commands and typeset since I'm not very familiar with and good at LaTex. After many times trying, I could successfully compose my Mathematics Autobiography. I learned that LaTex is a very good word processor for writing and composing mathematical documents. I will work more with my LaTex advisor and I believe that my skills in LaTex will soon be improved.

1.2 Python program

From the website of Dr. Martins, I downloaded and read Chapter 1 and Chapter 2 (section 2.7) of Python book. I also downloaded, installed Python and ran a demo program in Python. Now I’m reading how to create a Python function and trying to program simple something with Python (solving a quardic equation, for instance).

1.3 The Schrodinger Equation

I started reading the Schrodinger Equation but there were many points such as notations and some rules in Physics that I haven’t understood yet. They would be very good topics for my further researches.

2 Week of September 25th, 2007

This week, I read the paper "Tensor rank and the ill-posedness of the best low-rank approximation problem" (VIN DE SILVA and LEK-HENG LIM) and pointed out some definitions which were new to me:
- Tensor
- Rank of tensor
- Hyperdeterminant
- E-Y theorem
- Singular Value Decomposition

I also tried to give some examples for each definition, but I still not completely understand the tensors with order higher than 3. For the presentation week, I would talk about those things the may be the results #1 and #2.

3 Week of October 2th, 2007

Duy, Fan and I discussed about the Singular Value Decomposition (SVD), the E-Y theorem and their roles in the Tensor paper. We tried to give some specific examples for each of them as follow:

Singular Value Decomposition

\[
A = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{pmatrix}
\]

\[
U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.236 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.447 & 0 & 0 & 0 & 0.894 \\ 0 & 0 & 0 & 1 & 0 \\ -0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix}
\]

Then \( A = U.\Sigma.V^* \). Moreover, both \( U \) and \( V \) are orthogonal matrices.

\[
U.U^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
V^*,(V^*)^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.447 & 0 & 0 & 0 & 0.894 \\ 0 & 0 & 0 & 1 & 0 \\ -0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix} \begin{pmatrix} 0 & 0.447 & 0 & 0 & -0.894 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

We also discussed about the first two results. They are the most important ones of the tensor paper. In next week, we will present what we discussed this week.
4 Week of October 15th, 2007

4.1 Tensor

An order-k tensor $A \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$ can be simply considered as a vector in a k-dimensional space.

Example:

- $k = 0$: $A \in \mathbb{R}$ - a scalar
- $k = 1$: $A \in \mathbb{R}^{d_1}$ - a vector. $A = [x_1, x_2, \ldots, x_{d_1}]$ with index $A_i$
- $k = 2$: $A \in \mathbb{R}^{d_1 \times d_2}$ - a matrix.

\[
A = \begin{pmatrix}
1 & 2 & \ldots & d_2 \\
1 & 2 & \ldots & d_2 \\
\vdots & \vdots & \ddots & \vdots \\
d_1 & d_1 & \ldots & d_{d_1 d_2}
\end{pmatrix}
\]

with index $A_{ij}$

- $k = 3$: $A \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ - $A$ is a cubic with index $A_{ijk}$

![Fig 1: An 3-order tensor](image)

4.2 Tensor product

From page 8, we get the formal definition of tensor product. Since $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} \otimes \ldots \otimes \mathbb{R}^{d_k} \cong \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$, the tensor product can be understood as:

\[
x_1 \otimes x_2 \otimes \ldots \otimes x_k = [x_{(1)}^{(1)} \ldots x_{(k)}^{(k)}]_{j_1 \ldots j_k = 1}
\]

For example: $k = 3$, $d_1 = 4$, $d_2 = d_3 = 3$.

\[
\begin{pmatrix}
x_{(1)}^{(1)} \\
x_{(1)}^{(2)} \\
x_{(1)}^{(3)} \\
x_{(1)}^{(4)}
\end{pmatrix}
\begin{pmatrix}
x_{(2)}^{(1)} \\
x_{(2)}^{(2)} \\
x_{(2)}^{(3)} \\
x_{(2)}^{(4)}
\end{pmatrix}
\begin{pmatrix}
x_{(3)}^{(1)} \\
x_{(3)}^{(2)} \\
x_{(3)}^{(3)} \\
x_{(3)}^{(4)}
\end{pmatrix}
\]

then

\[
x_1 \otimes x_2 \otimes \ldots \otimes x_k = [x_{(1)}^{(1)}x_{(2)}^{(2)}x_{(3)}^{(3)}x_{(1)}^{(1)}x_{(2)}^{(2)}x_{(3)}^{(3)}x_{(1)}^{(1)}x_{(2)}^{(2)}x_{(3)}^{(3)}x_{(1)}^{(1)}x_{(2)}^{(2)}x_{(3)}^{(3)}]
\]
A tensor $A \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$ is said to be decomposable if it can be written in the form

$$A = x_1 \otimes x_2 \otimes \ldots \otimes x_k$$

where $x_i \in \mathbb{R}^{d_i}$ for all $i = 1,\ldots, k$.

With matrices $(L_1, \ldots, L_k).x_1 \otimes x_2 \otimes \ldots \otimes x_k = L_1 x_1 \otimes L_2 x_2 \otimes \ldots \otimes L_k x_k$

### 4.3 Some results of the paper

Best rank-r approximation to a tensor $A \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$

$$||A - x_1 \otimes y_1 \otimes \ldots \otimes z_1 - \ldots - x_r \otimes y_r \otimes \ldots \otimes z_r||$$

or \( \text{argmin}_{\text{rank} \leq r} ||A - B|| \) (APPROX(A,r))

**Result 1**

APPROX(A,r) is ill-posed for many r. We will show that, regardless of the choice of norm, the problem of determining a best rank-r approximation for an order-k tensor in $\mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$ has no solution in general for $r = 2, \min \{d_1, \ldots, d_k\}$ and $k \geq 3$. In other words, the best low rank approximation problem for tensors is ill-posed for all orders (higher than 2), all norms, and many ranks.

**Result 2**

APPROX(A,r) is ill-posed for many A. We will show that the set of tensors that fail to have a best low rank approximation has positive volume. In other words, such failures are not rare if one randomly picks a tensor $A$ in a suitable tensor space, then there is a non-zero probability that $A$ will fail to have a best rank-r approximation for some $r < \text{rank}(A)$.

**Result 3**

Weak solutions to approx(A,r). We will propose a natural way to overcome the ill-posedness of the best rank-r approximation problem with the introduction of weak solutions and we characterize all weak solutions in the case $r = 2, k = 3$.

**Result 4**

Semialgebraic description of tensor rank. We will show that for any $d_1, \ldots, d_k$, there exists a finite number of polynomial functions $\Delta_1, \ldots, \Delta_m$, defined on $\mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$ that the rank of any $A \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$ is completely determined by the signs of $\Delta_1(A), \ldots, \Delta_m(A)$. We work out in the special case $\mathbb{R}^2 \times 2 \times 2$.

**Result 5**

Reduction. We will give techniques for reducing certain questions about tensors (orbits, invariant, limits) from high-dimensional tensor spaces to lower-
dimensional tensor spaces. For instance, if two tensors in $\mathbb{R}^{c_1 \times c_2 \times \cdots \times c_k}$ lie in distinct $GL_{c_1, c_2, \ldots, c_k}(\mathbb{R})$-orbits, then they lie in distinct $GL_{d_1, d_2, \ldots, d_k}(\mathbb{R})$-orbits in $\mathbb{R}^{d_1 \times d_2 \times \cdots \times d_k}$ for any $d_i \geq c_i$.

5  Week of October 22\textsuperscript{th}, 2007

Last week I read Theorem 1.1, 1.2 and 1.3. I also tried section 2.1 and section 2.2 but there was some things that I was not completely understand yet

1. $GL_{d_1, \ldots, d_k}$
2. $O_{d_1, \ldots, d_k}$

In this week, I will try to complete section 2.2 and continue reading section 2.3 and programming some simple things with Python.