score	possible	page
	20	1
	30	2
	20	3
	30	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/20 1. Set up and evaluate the double integral to find the volume between $f_1(x,y) = \sin(x)\cos(y)$ and $f_2(x,y) = \cos(x)\sin(y) + 2$ over the triangle with corners $(0,0)$, $(\pi,0)$, and (π,π).

[14.6 #7]

$$\begin{aligned}
\int_0^\pi \int_0^x (f_2(x,y) - f_1(x,y))\,dydx &= \int_0^\pi \int_0^x (\cos(x)\sin(y) + 2 - \sin(x)\cos(y))\,dydx \\
&= \int_0^\pi -\cos(x)\cos(y) + 2y - \sin(x)\sin(y)\big|_0^x\,dx \\
&= \int_0^\pi (-\cos(x)\cos(x) + 2x - \sin(x)\sin(x)) - (-\cos(x)\cos(0) + 0 - \sin(x)\sin(0))\,dx \\
&= \int_0^\pi -1 + 2x + \cos(x)dx = -x + x^2 + \sin(x)\big|_0^\pi = -\pi + \pi^2 + 0 - 0 = \pi^2 - \pi\,.
\end{aligned}$$

/10 2. Set up the iterated integral to compute the surface area of the graph of $f(x, y) = x^2 - y^3$ over the rectangle with opposite corners $(2, 3)$ and $(5, 7)$. (You do not need to compute the integral.)

[similar to 14.5 #9] Computing $f_x(x, y) = 2x$ and $f_y(x, y) = -3y^2$, the surface area is

$$\iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA = \int_2^5 \int_3^7 \sqrt{1 + (2x)^2 + (-3y^2)^2} dydx\,.$$

/20 3. Using spherical coordinates, find the mass of a ball of radius 2 centered at the origin with density function $\delta(x, y, z) = x^2 + y^2 + z^2$.

[similar to 14.7 #33] Recalling that $\rho = \sqrt{x^2 + y^2 + z^2}$, the density simplifies to $\delta = \rho^2$. Since the region is a ball of radius 2, we have $\phi \in [-\pi/2, \pi/2]$, $\theta \in [0, 2\pi]$, and $\rho \in [0, 2]$. Recalling that $dV = \rho^2 \cos(\phi) d\rho d\theta d\phi$, our integral is thus

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2)\; dV &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^4 \cos(\phi) d\rho d\theta d\phi \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \frac{\rho^5}{5} \cos(\phi)\Big|_0^2 d\theta d\phi = \frac{2^5}{5} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \cos(\phi) d\theta d\phi \\ &= \frac{2^6\pi}{5} \int_{-\pi/2}^{\pi/2} \cos(\phi) d\phi = \frac{2^6\pi}{5} (\sin(\pi/2) - \sin(-\pi/2)) = \frac{2^7\pi}{5}\end{aligned}$$

/20 4. Use the transformation $y - x = u$ and $x + y = v$ to evaluate

$$\iint_R \sin(x-y)dA$$

on the square R determined by the lines $y = x$, $y = -x + 2$, $y = x + 2$, and $y = -x$.
(R has corners $(0,0)$, $(1,1)$, $(-1,1)$, and $(0,2)$.)

[openstax 5.7 #393] Solving the system $y - x = u$ and $x + y = v$ for x and y gives $x = (v-u)/2$ and $y = (u+v)/2$, so the Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{-1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{2} = -\frac{1}{2}$$

and the integrand is

$$\sin(x-y) = \sin((v-u)/2 - (u+v)/2) = \sin(-u)\,.$$

The transformation is linear and the corners go to $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$, so S is the square $0 \le u \le 2$ and $0 \le v \le 2$. Putting this all together, we have

$$\iint_R \sin(x-y)dA = \int_0^2 \int_0^2 \sin(-u)\left|-\frac{1}{2}\right| du dv = \frac{1}{2}\int_0^2 \cos(-u)|_0^2\, dv = \frac{1}{2}\int_0^2 \cos(-2) - 1 dv = \cos(-2) - 1\,.$$

5. The parameterized curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \leq t \leq 4\pi$ follows a thin wire with density $\delta(x, y, z) = z$.

[15.1 #19]

/10 (a) Find the mass (M) of this wire.

From the parameterization, we compute

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle, \quad \text{so}$$
$$\|\vec{r}'(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}\,.$$

Noting $\delta(x, y, z) = z = t$,

$$M = \int_0^{4\pi} t\sqrt{2}dt = \frac{t^2}{2}\sqrt{2}\bigg|_0^{4\pi} = \frac{(4\pi)^2}{2}\sqrt{2} = 8\sqrt{2}\pi^2.$$

/6 (b) Write the integrals for the three moments of this wire. (You do not need to compute the integrals.)

$$M_{yz} = \int_0^{4\pi} \cos(t)t\sqrt{2}dt$$

$$M_{xz} = \int_0^{4\pi} \sin(t)t\sqrt{2}dt$$

$$M_{xy} = \int_0^{4\pi} tt\sqrt{2}dt$$

/4 (c) Write the expression for the center of mass of the wire in terms of M, M_{yz}, M_{xz}, and M_{xy}.

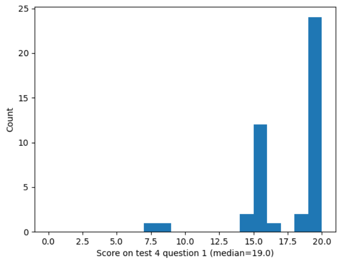
$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right).$$

/10 6. Compute the divergence and curl of the vector field $\vec{F} = \langle x^2 + z^3, x^5 + y^7, y^9 + z \rangle$.

[similar to 15.2 #14]

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = 2x + 7y^6 + 1$$
$$\text{curl}\vec{F} = \nabla \times \vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle = \langle 9y^8 - 0, 3z^2 - 0, 5x^4 - 0 \rangle = \langle 9y^8, 3z^2, 5x^4 \rangle$$

Scores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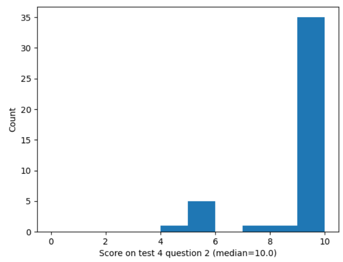
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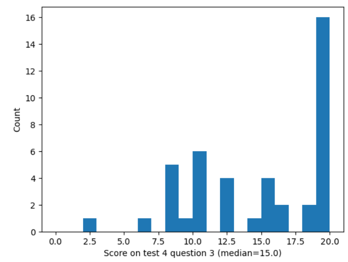
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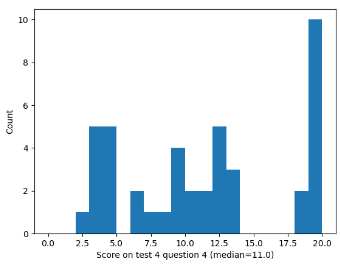
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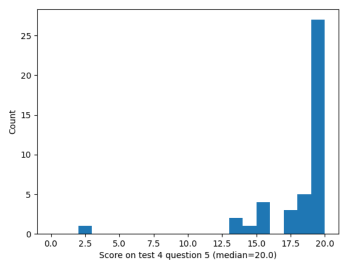
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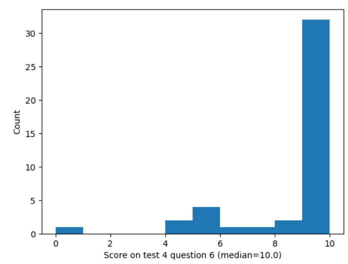
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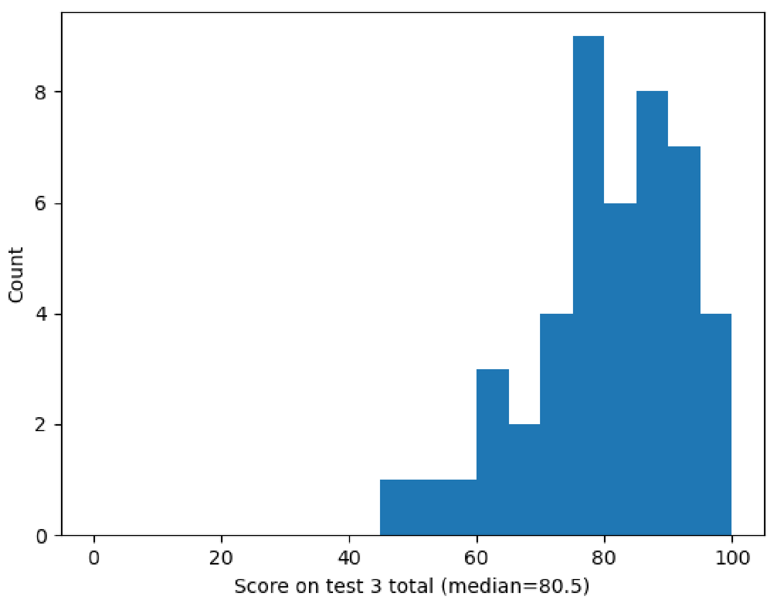
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Score on test 3 total (median=80.5)