

score	possible	page
	25	1
	25	2
	25	3
	25	4
	100	

Name: _____

Show your work!

You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

- /5 1. (a) Given that $z = 5x + 2y$, $x = 2\cos(t) + 1$, and $y = \sin(t) - 3$, find the value(s) of t where $\frac{dz}{dt} = 0$. (Since you do not have a calculator, leave your answer as some expression $t = \dots$) [13.5 #15]

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 5(-2\sin(t)) + 2(\cos(t)) \stackrel{\text{set}}{=} 0 \Rightarrow \tan(t) = \frac{1}{5} \Leftrightarrow t = \arctan(1/5).$$

- /5 (b) Given that $\frac{\partial z}{\partial x} = 2$, $\frac{\partial z}{\partial y} = 3$, $\frac{\partial x}{\partial s} = 5$, $\frac{\partial x}{\partial t} = 7$, $\frac{\partial y}{\partial s} = 11$, $\frac{\partial y}{\partial t} = 13$, find $\frac{\partial z}{\partial t}$. [similar to 13.5 #30]

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 2 \cdot 7 + 3 \cdot 13 = 14 + 39 = 53.$$

2. Let $f(x, y) = -x^2y + xy^2 + xy$ and $P = (2, 1)$. [similar to 13.6 #19]

- /5 (a) Find the direction of maximal increase of f at P .

$$\nabla f(x, y) = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle \Rightarrow \nabla f(2, 1) = \langle -4 + 1 + 1, -4 + 4 + 2 \rangle = \langle -2, 2 \rangle.$$

- /5 (b) What is the maximal value of $D_{\vec{u}}f$ at P , where \vec{u} is a unit vector?

$$\|\nabla f(2, 1)\| = \| \langle -2, 2 \rangle \| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}.$$

- /5 (c) Give a unit vector \vec{u} such that $D_{\vec{u}}f = 0$ at P . We need $\vec{u} \cdot \nabla f(2, 1) = \vec{u} \cdot \langle -2, 2 \rangle = 0$, which is satisfied by any nonzero multiple of $\langle 1, 1 \rangle$. Dividing by its norm, the two possible unit vectors are $\vec{u} = \pm \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$.

3. Consider the surface $2x^2 + y^2 + 3z^2 = 10$ and the point $P = (-1, 1, 2)$. [similar to 13.7 #21]

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- (a) Find an equation for the normal line to the surface at P . Consider the surface as the level surface of a function $f(x, y, z)$ and compute the gradient

$$\nabla f(x, y, z) = \langle 4x, 2y, 6z \rangle.$$

The gradient is orthogonal to the surface, so an equation for the normal line is

$$l(t) = \langle -1, 1, 2 \rangle + t\nabla f(-1, 1, 2) = \langle -1, 1, 2 \rangle + t\langle -4, 2, 12 \rangle.$$

(Note: The 10 in the problem statement should have been 15, so that the point was on the surface.)

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- (b) Find an equation for the plane tangent to the surface at P . The gradient and point also give an equation for the tangent plane

$$\nabla f(-1, 1, 2) \cdot \overrightarrow{P(x, y, z)} = -4(x - (-1)) + 2(y - 1) + 12(z - 2) = 0.$$

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4. Find the critical points of the function $f(x, y) = x^3y + 12x^2 - 8y$. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point. [similar to 13.8 #12] Compute the gradient

$$\nabla f(x, y) = \langle 3x^2y + 24x, x^3 - 8 \rangle$$

and set equal to $\langle 0, 0 \rangle$. The second component gives $x^3 - 8 = 0$ so $x = 2$. Plugging into the first component gives $12y + 48 = 0$ so $y = -4$. The second partial derivatives are

$$f_{xx}(x, y) = 6xy + 24, \quad f_{yy}(x, y) = 0, \quad \text{and} \quad f_{xy}(x, y) = 3x^2,$$

so

$$D(x, y) = (6xy + 24)(0) - (3x^2)^2 \quad \text{and} \quad D(2, -4) = -(3(2^2))^2 < 0,$$

which means there is a saddle point at $(2, -4)$.

- /15 5. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = y^2 - x^2$ subject to the constraint $\frac{1}{4}x^2 + y^2 = 1$. [Similar to openstax 4.8 #365] Setting $g(x, y) = \frac{1}{4}x^2 + y^2 - 1$, we have

$$\nabla f(x, y) = \langle -2x, 2y \rangle \text{ and } \nabla g(x, y) = \left\langle \frac{1}{2}x, 2y \right\rangle,$$

so our equations are

$$-2x = \lambda \frac{1}{2}x, \quad 2y = \lambda 2y, \quad \text{and } \frac{1}{4}x^2 + y^2 - 1 = 0.$$

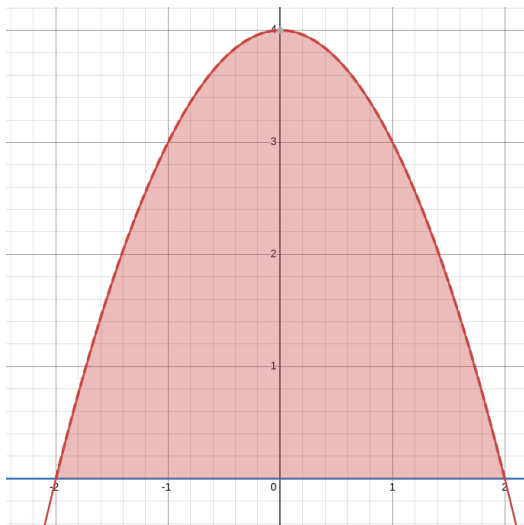
The first equation implies either $\lambda = -4$ or $x = 0$. If $\lambda = -4$, then the second equation implies $y = 0$ and then the third equation implies $x = \pm 2$; thus we have candidate points $(2, 0)$ and $(-2, 0)$. If $x = 0$, then the second equation implies $\lambda = 1$ and the third equation implies $y = \pm 1$; thus we have candidate points $(0, 1)$ and $(0, -1)$. Plugging in these 4 points gives

$$f(2, 0) = -4, \quad f(-2, 0) = -4, \quad f(0, 1) = 1, \quad \text{and } f(0, -1) = 1.$$

So, the maximum value is 1 and the minimum value is -4.

6. Consider the iterated integral $\int_{-2}^2 \int_0^{4-x^2} dy dx$. (You do not need to compute the integral.) [14.1 #17]

- /5 (a) Sketch the region whose area it computes.



- /5 (b) Switch the order of integration to give another iterated integral that computes the same area. (You do not need to compute the integral.)

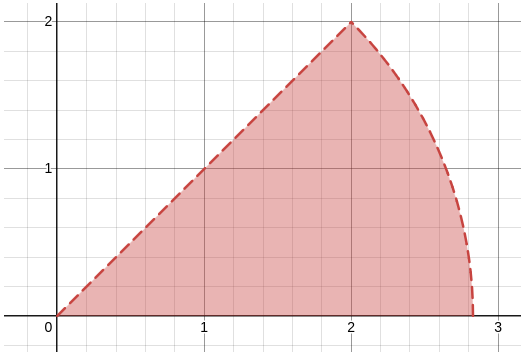
$$\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx dy$$

- /15 7. Compute the iterated integral $\int_1^2 \int_y^3 (x^2y - \cos(x)) dx dy$. (You do not need to simplify your answer.)
[similar to 14.2 #8]

$$\begin{aligned} &= \int_1^2 \left(\frac{1}{3}x^3y - \sin(x) \right) \Big|_y^3 dy = \int_1^2 \left((9y - \sin(3)) - \left(\frac{1}{3}y^4 - \sin(y) \right) \right) dy \\ &= \int_1^2 \left(9y - \sin(3) - \frac{1}{3}y^4 + \sin(y) \right) dy = \frac{9}{2}y^2 - \sin(3)y - \frac{1}{15}y^5 - \cos(y) \Big|_1^2 \\ &= \frac{9}{2}2^2 - \sin(3)2 - \frac{1}{15}2^5 - \cos(2) - \left(\frac{9}{2} - \sin(3) - \frac{1}{15} - \cos(1) \right) \end{aligned}$$

8. Consider the integral $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y) dx dy$. (You do not need to compute the integral.) [similar to 14.3 #13]

- /5 (a) Sketch the region that is being integrated over.



- /5 (b) Rewrite the integral in polar coordinates. (You do not need to compute the integral.) The region is a sector of a disc, with $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq \sqrt{8}$. We thus get

$$\int_0^{\pi/4} \int_0^{\sqrt{8}} (r \cos(\theta) + r \sin(\theta)) r dr d\theta$$

Scores

