score	possible	page
	20	1
	30	2
	25	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/4 1. (a) Complete the definition of *The Limit of a Function f at a point.*:
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The statement that "the limit of $f(x)$, as x approaches c, is L" is denoted by

$$\lim_{x\to c} f(x) = L,$$

and means that
given any $\varepsilon > 0$,
there exists $\delta > 0$ such that for all x in I, where $x \neq c$,
if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

/6 (b) Use this definition to prove that $\lim_{x\to 3} 5x + 4 = 19$.
Starting from $|(5x + 4) - 19| = |f(x) - L| < \varepsilon$, we can simplify to $|5x - 15| < \varepsilon$ and then divide by 5 to get $|x - 3| < \varepsilon/5$. Given any $\varepsilon > 0$, we can choose $\delta = \varepsilon/5$ and have

$$|x - 3| < \delta = \varepsilon/5 \Leftrightarrow |5x - 15| < \varepsilon \Leftrightarrow |(5x + 4) - 19| = |f(x) - L| < \varepsilon\,,$$

so we have proven the limit.

/4 2. (a) Complete the statement of *The Squeeze Theorem*:
Let f, g and h be functions on an open interval I containing c such that for all x in I, (except possibly at $x = c$)

$$f(x) \le g(x) \le h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

/6 (b) Use the Squeeze Theorem to evaluate $\lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right)$.
Set

$$f(x) = -2|x|\,,$$
$$g(x) = 2x\sin\left(\frac{1}{x}\right)\,, \quad \text{and}$$
$$h(x) = 2|x|\,.$$

Since $-1 \le \sin(\cdot) \le 1$, we have $f(x) \le g(x) \le h(x)$ when x is near 0 (and for all $x \neq 0$), so the first assumption of the Squeeze Theorem holds with $c = 0$.
We can compute $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so the second assumption of the Squeeze Theorem also holds with $c = 0$ and $L = 0$.
Thus the conclusion of the Squeeze Theorem holds and we can conclude

$$\lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) = \lim_{x\to c} g(x) = L = 0\,.$$

/4 3. (a) Complete the definition of the *Derivative Function*:

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is the derivative of f.

/6 (b) Using this definition, compute the derivative of $f(x) = 3x^2 + 7x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &&= \lim_{h\to 0} \frac{(3(x+h)^2 + 7(x+h)) - (3x^2+7x)}{h} \\ &= \lim_{h\to 0} \frac{(3(x^2+2xh+h^2)+7x+7h)-(3x^2+7x)}{h} &&= \lim_{h\to 0} \frac{3x^2+6xh+3h^2+7x+7h-3x^2-7x}{h} \\ &= \lim_{h\to 0} \frac{6xh+3h^2+7h}{h} &&= \lim_{h\to 0} \frac{h(6x+3h+7)}{h} \\ &= \lim_{h\to 0}(6x+3h+7) &&= 6x+7\,. \end{aligned}$$

/4 4. (a) Complete the statement of the *Intermediate Value Theorem*:

Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$.
Then for every value y, where $f(a) < y < f(b)$,
there is at least one value c in (a, b) such that $f(c) = y$

/6 (b) Use the Intermediate Value Theorem to show that the equation $x^7 + x^2 = 4$ has a solution.

Let $f(x) = x^7 + x^2$, so we want to show a solution to $f(x) = 4$ exists. Since x^7 and x^2 are both continuous, so is $f(x)$. Plugging in, we find

$$f(0) = 0 + 0 = 0 < 4 \quad \text{and}$$
$$f(2) = 2^7 + 2^2 = 2^7 + 4 > 4\,.$$

So, by the Intermediate Value Theorem with $a = 0$, $b = 2$, and $y = 4$, there must exist $0 < c < 2$ such that $f(c) = 4$.

5. Compute the following limits. Use properties of limits and algebra, not the ε-δ definition. Do **not** use L'Hôpital's rule.

/4 (a) $\displaystyle\lim_{x\to 4^+} \frac{x+3}{x-4} =$

Since $x \to 4^+$, we know $x - 4 > 0$, so we have

$$\lim_{x\to 4^+} \frac{7}{x-4} = \lim_{t\to 0^+} \frac{7}{t} = \infty\,.$$

/6 (b) $\displaystyle\lim_{x\to 0} \frac{4^{-1} - (4-x)^{-1}}{x} =$

$$\lim_{x\to 0} \frac{\frac{1}{4} - \frac{1}{4-x}}{x} = \lim_{x\to 0} \frac{\frac{4-x-4}{4(4-x)}}{x} = \lim_{x\to 0} \frac{\frac{-x}{4(4-x)}}{x} = \lim_{x\to 0} \frac{-x}{x4(4-x)} = \lim_{x\to 0} \frac{-1}{4(4-x)} = \frac{-1}{4(4-0)} = \frac{-1}{16}$$

6. Compute the following derivatives. (Use derivative rules, rather than computing the limits.)

/2 (a) $f(x) = \cot(x) \Rightarrow f'(x) = -\csc^2(x)$

/2 (b) $f(x) = \sec(x) \Rightarrow f'(x) = \sec(x)\tan(x)$

/2 (c) $f(x) = \csc(x) \Rightarrow f'(x) = -\csc(x)\cot(x)$

/2 (d) $f(x) = \dfrac{1}{x^2} \Rightarrow f'(x) = -2x^{-3}$

/2 (e) $f(x) = 9^x \Rightarrow f'(x) = 9^x \ln(9)$

/2 (f) $f(x) = \log_2(x) \Rightarrow f'(x) = \dfrac{1}{x\ln(2)}$

/4 (g) $\dfrac{d}{dx}\left(3x^4 + \cos(x)\right)^9 =$

$$9\left(3x^4+\cos(x)\right)^8\left(\frac{d}{dx}\left(3x^4+\cos(x)\right)\right) = 9\left(3x^4+\cos(x)\right)^8\left(3(4x^3)-\sin(x)\right)$$

/9 (h) $\dfrac{d}{dt}\left(7\sin\left(\dfrac{\cos(t)}{1+4^t}\right)(5t^4+3t)\right) =$

$$\begin{aligned}
&7\left(\left(\frac{d}{dt}\sin\left(\frac{\cos(t)}{1+4^t}\right)\right)(5t^4+3t)+\sin\left(\frac{\cos(t)}{1+4^t}\right)\left(\frac{d}{dt}(5t^4+3t)\right)\right)\\
=&7\left(\cos\left(\frac{\cos(t)}{1+4^t}\right)\left(\frac{d}{dt}\left(\frac{\cos(t)}{1+4^t}\right)\right)(5t^4+3t)+\sin\left(\frac{\cos(t)}{1+4^t}\right)(5(4t^3)+3)\right)\\
=&7\left(\cos\left(\frac{\cos(t)}{1+4^t}\right)\frac{(-\sin(t))(1+4^t)-\cos(t)(4^t\ln(4))}{(1+4^t)^2}(5t^4+3t)+\sin\left(\frac{\cos(t)}{1+4^t}\right)(5(4t^3)+3)\right)
\end{aligned}$$

7. Sketch the graph of a single function f that satisfies the following conditions.

/1 (a) f has domain $[0, 12)$.

/1 (b) $f(x) < 4$ always.

/1 (c) $f(1) = 2$

/1 (d) $\lim_{x\to 2^-} f(x) = 1$.

/1 (e) $\lim_{x\to 2^+} f(x) = 3$.

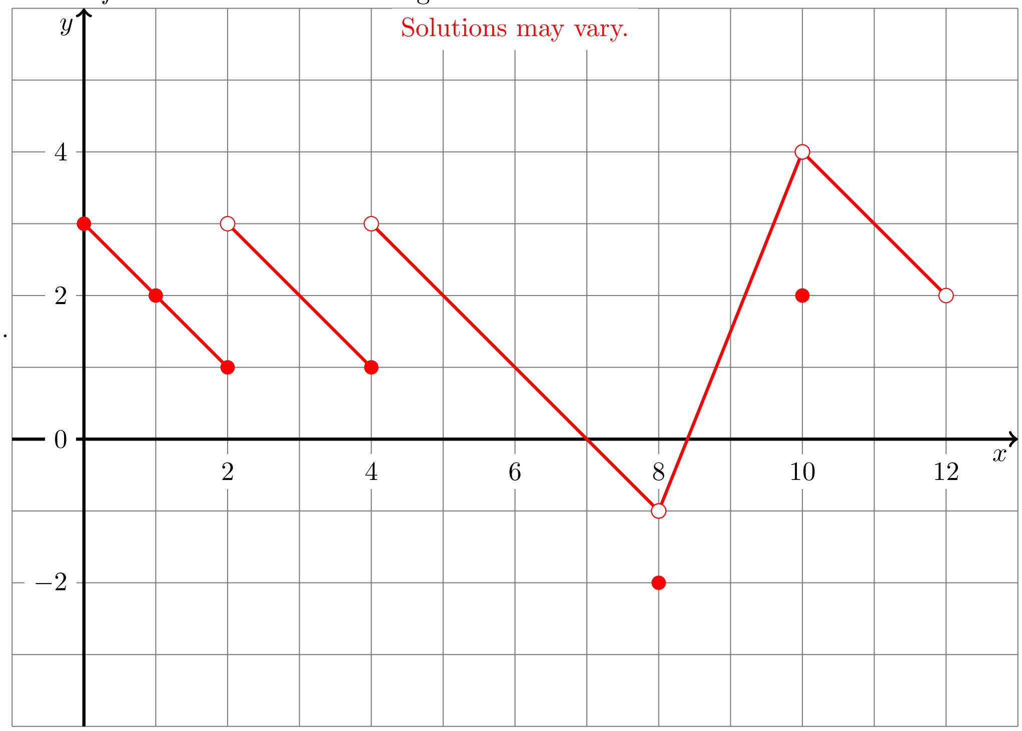
/1 (f) $\lim_{x\to 4} f(x)$ does not exist.

/1 (g) $\lim_{x\to 8} f(x) = -1$.

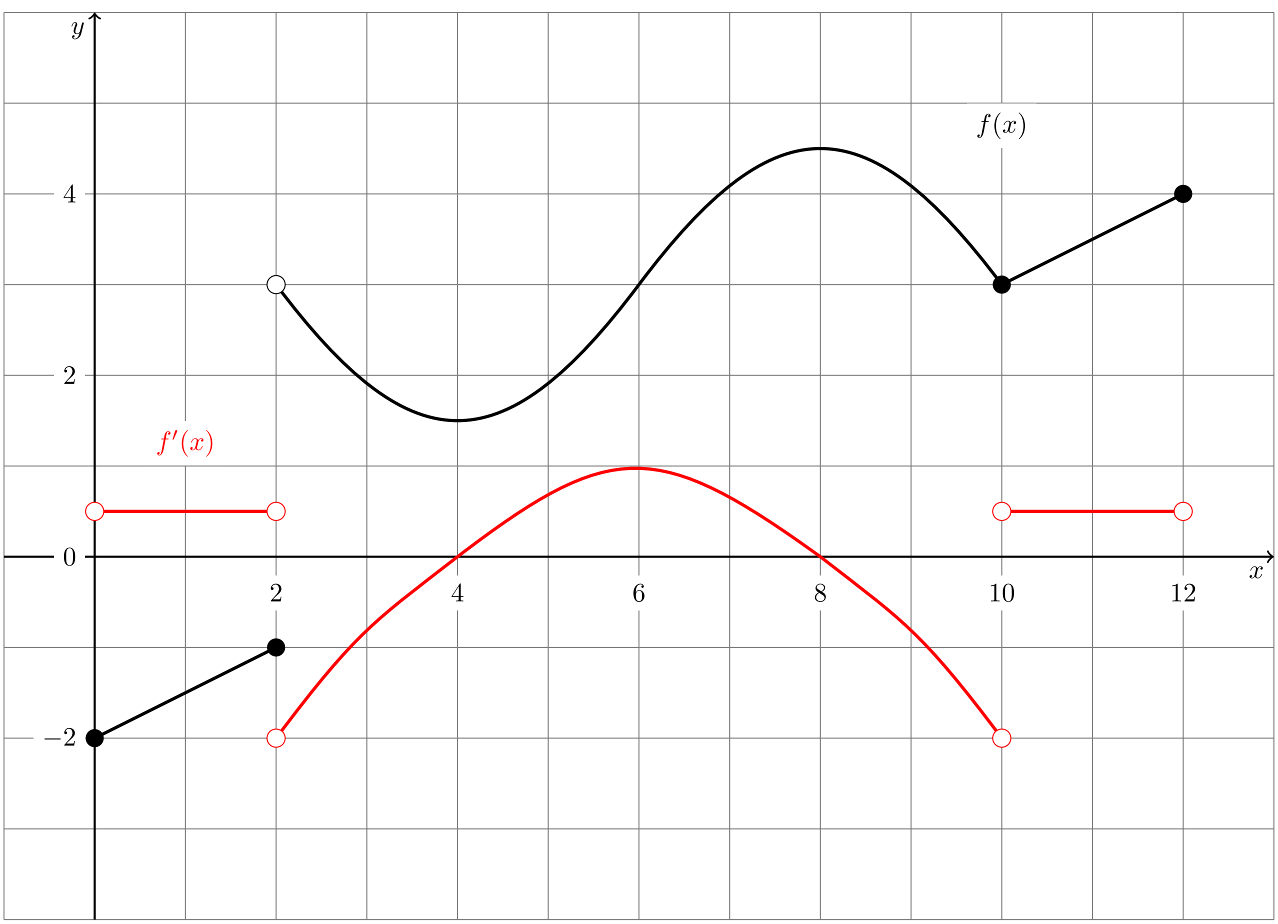
/1 (h) $f(8) = -2$

/1 (i) $\lim_{x\to 10} f(x) = 4$.

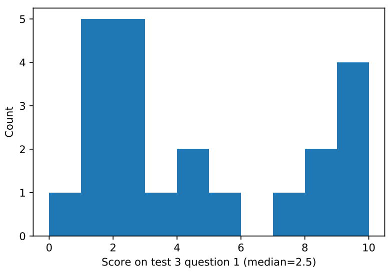
/1 (j) $\lim_{x\to 12^-} f(x) = 2$.



/15 8. The graph of a function f is given below. On the same axes, sketch the graph of f'.



Scores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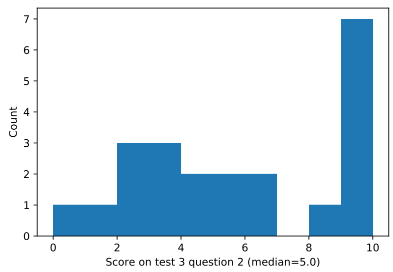
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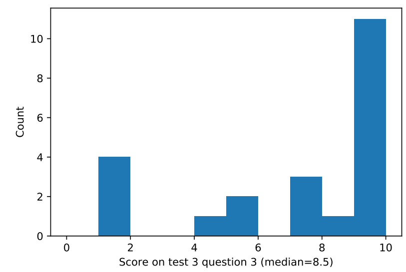
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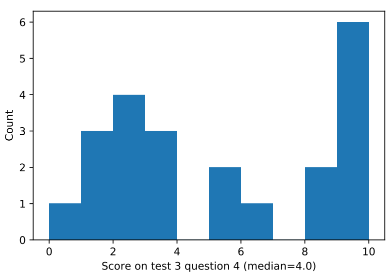
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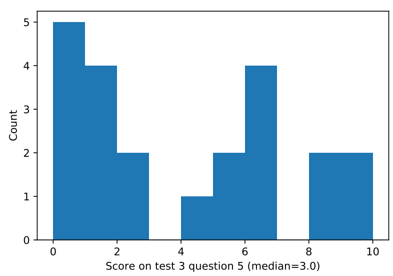
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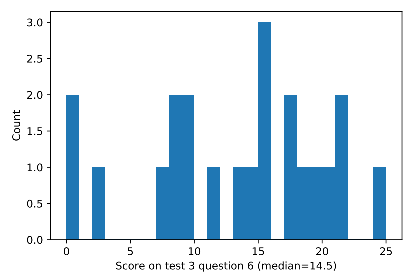
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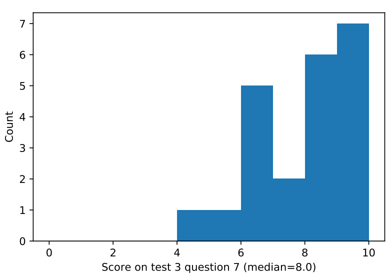
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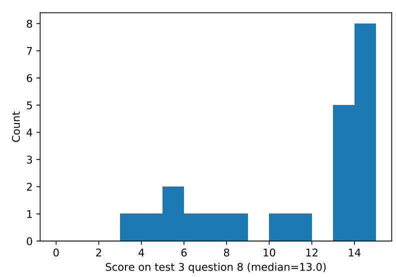
Score on test 3 question 7 (median=8.0)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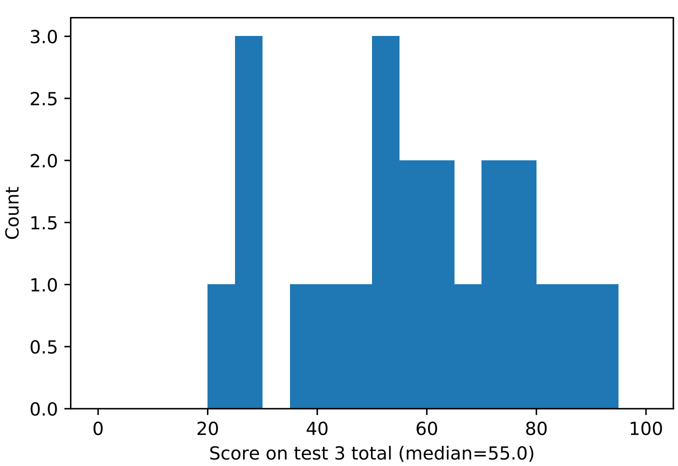
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