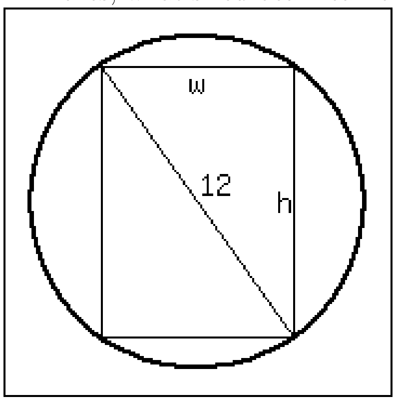
score	possible	page
	20	1
	30	2
	28	3
	22	4
	100	

Name: ________________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/20 1. [4.3 # 13] The strength S of a wooden beam is directly proportional to its cross sectional width w and the square of its height h. That is, $S = kwh^2$ for some constant k. Given a circular log with diameter of 12 inches, what sized beam can be cut from the log with maximum strength?



Since the diameter is 12, we have the constraint $w^2 + h^2 = 12^2$. Solving for h^2 gives $h^2 = 12^2 - w^2$ and then substituting in S gives

$$S = kwh^2 = kw(12^2 - w^2) = 12^2kw - kw^3\,.$$

Differentiating with respect to w gives

$$S' = 12^2k - 3kw^2\,,$$

which always exists. Setting equal to 0 and solving for w gives critical values

$$12^2k - 3kw^2 = 0 \quad \Leftrightarrow \quad w^2 = \frac{12^2}{3} \quad \Leftrightarrow \quad w = \pm\frac{12}{\sqrt{3}}\,.$$

We can ignore the negative solution.

To show this critical value gives a maximum, there are several methods that would work:

- The domain for w is $[0, 12]$ since w and h cannot be negative. Since $S = 0$ when $w = 0$ or $w = 12$ and

$$S = k\frac{12}{\sqrt{3}}\left(12^2 - \left(\frac{12}{\sqrt{3}}\right)^2\right) = k\frac{12^3}{\sqrt{3}}\frac{2}{3} > 0$$

  at the critical value, that gives the maximum.
- Using the first derivative test, we can select test points and see $S'(1) = k(12^2 - 3) > 0$ and $S'(11) = k(12^2 - 3 \cdot 11^2) < 0$, so the critical value gives a local maximum.
- Differentiating S' again yields $S'' = -6kw$. Since this is negative when w is positive (and k is positive), we know our solution is at a local maximum by the second derivative test.

Our best dimensions are

$$w = \frac{12}{\sqrt{3}} = 4\sqrt{3} \quad \text{and}$$

$$h = \sqrt{12^2 - w^2} = \sqrt{12^2 - \left(\frac{12}{\sqrt{3}}\right)^2} = 12\sqrt{1 - \frac{1}{3}} = 12\sqrt{\frac{2}{3}} = 4\sqrt{6}\,.$$

/10 2. [4.1 #5 shortened] Perform 2 iterations of Newton's method to the function $f(x) = x^2 + x - 2$ starting from $x_0 = 0$.

Computing $f'(x) = 2x + 1$, the Newton's update is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 + x_n - 2}{2x_n + 1},$$

so

$$\begin{aligned} x_1 &= x_0 - \frac{x_0^2 + x_0 - 2}{2x_0 + 1} = 0 - \frac{0^2 + 0 - 2}{0 + 1} = -\frac{-2}{1} = 2 \\ x_2 &= x_1 - \frac{x_1^2 + x_1 - 2}{2x_1 + 1} = 2 - \frac{2^2 + 2 - 2}{2(2) + 1} = 2 - \frac{4}{5} = 6/5\,. \end{aligned}$$

/20 3. [4.2 #15] A company that produces landscaping materials is dumping sand into a conical pile. The sand is being poured at a rate of $7\,\text{ft}^3/\text{s}$. The physical properties of the sand, in conjunction with gravity, ensure that the cone's height is roughly 5/6 the length of the diameter of the circular base.

How fast is the cone rising when it has a height of 30 feet?

The volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius and h is the height. We are told $\frac{dV}{dt} = 7\,\text{ft}^3/\text{s}$ and are asked to find $\frac{dh}{dt}$ when $h = 30\,\text{ft}$. We are told $h = \frac{5}{6}2r = \frac{5}{3}r$, so $r = \frac{3}{5}h$. Substituting, we have

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3}{25}\pi h^3\,.$$

Differentiating with respect to t gives

$$\frac{dV}{dt} = \frac{3}{25}\pi 3h^2\frac{dh}{dt}\,.$$

Solving for $\frac{dh}{dt}$ and substituting in the known values gives

$$\frac{dh}{dt} = \frac{dV}{dt}\frac{25}{9\pi h^2} = 7\frac{25}{9\pi 30^2}\ \text{ft/s}\,.$$

/9 4. [4.4 # 11] Use differentials to approximate $\sqrt{49.6}$.

Setting $f(x) = \sqrt{x}$, the differential is $dy = f'(x)dx = \frac{1}{2\sqrt{x}}dx$. Using the base point $x = 49$, we have $dx = 0.6$ and

$$\sqrt{49.6} = f(49 + dx) \approx f(49) + dy = \sqrt{49} + \frac{1}{2\sqrt{49}}0.6 = 7 + \frac{1}{14}0.6 = 7 + \frac{3}{70}\,.$$

/9 5. [4.4 # 31] A set of plastic spheres are to be made with a diameter of 1 cm. If the manufacturing process is accurate to 1 mm, what is the propagated error in volume of the spheres?

The volume of the sphere is $V = \frac{4}{3}\pi r^3$ so the differential is $dV = 4\pi r^2 dr$. Converting to millimeters, we have $r = 5$ and $dr = 1/2$, so

$$dV = 4\pi r^2 dr = 4\pi 5^2 \frac{1}{2} = 50\pi\,,$$

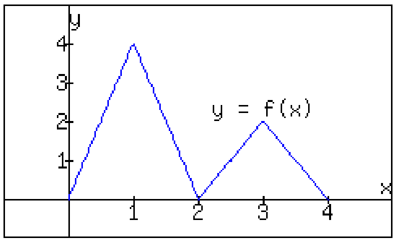
measured in mm^3.

/10 6. [5.1 # 37] Find the function $f(x)$ that has $f''(x) = 2e^x$, $f'(0) = -6$, and $f(0) = -8$.

Antidifferentiating once, we have $f'(x) = 2e^x + C$. Since $f'(0) = -6$, we have $-6 = 2e^0 + C$, so $C = -6 - 2 = -8$ and $f'(x) = 2e^x - 8$.

Antidifferentiating again, we have $f(x) = 2e^x - 8x + D$. Since $f(0) = -8$, we have $-8 = 2e^0 - 8(0) + D$, so $D = -8 - 2 = -10$ and $f(x) = 2e^x - 8x - 10$.

7. [5.2 # 7] A graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.



/2 (a) $\int_0^2 f(x)\,dx \;= \frac{1}{2}2\cdot 4 = 4$

/2 (b) $\int_2^4 f(x)\,dx \;= \frac{1}{2}2\cdot 2 = 2$

/2 (c) $\int_2^4 2f(x)\,dx \;= 2\int_2^4 2f(x)\,dx = 4$

/2 (d) $\int_0^1 4x\,dx \;= \int_0^1 f(x)\,dx = \frac{1}{2}1\cdot 4 = 2$

/2 (e) $\int_2^3 (2x-4)\,dx = \int_2^3 f(x)\,dx = \frac{1}{2}1\cdot 2 = 1$

/2 (f) $\int_2^3 (4x-8)\,dx \;= 2\int_2^3 f(x)\,dx = 2$

8. [5.3 # 33 rephrased] Consider the integral $\displaystyle\int_a^b f(x)dx = \int_1^2 \ln(x)dx$.

/2 (a) Graph $f(x) = \ln(x)$ on the interval $[a,b] = [1,2]$.

/4 (b) Add to the sketch the rectangles used to approximate the integral using the Midpoint Rule with 3 rectangles.

/4 (c) Approximate the integral by summing the areas of the rectangles. (Do not simplify.)

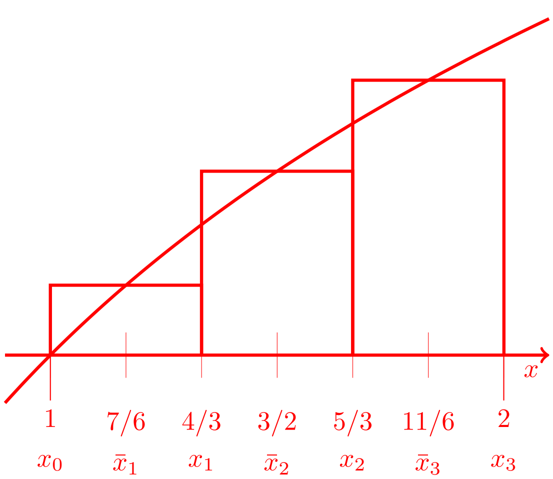
The function $\ln(x)$ is zero at $x = 1$ and then is increasing and concave down.

The interval $[a,b] = [1,2]$ has length 1 and we are using 3 rectangles, so the width of each rectangle is $\Delta x = 1/3$.

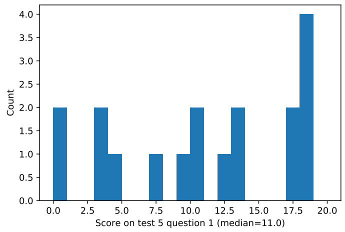
The base of the first rectangle is $[1, 1+1/3]$, which has midpoint $\bar{x}_1 = 1 + 1/6 = 7/6$. The second has base $[1+1/3, 1+2/3]$ and midpoint $\bar{x}_2 = 3/2$, and the third has base $[1+2/3, 2]$ and midpoint $\bar{x}_3 = 1+5/6 = 11/6$.

The approximation of the integral is

$$\sum_{i=1}^{3} f(\bar{x}_i)\Delta x = \left(\ln(\bar{x}_1) + \ln(\bar{x}_2) + \ln(\bar{x}_3)\right)\Delta x = \left(\ln\left(\frac{7}{6}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{11}{6}\right)\right)\frac{1}{3}.$$



Scores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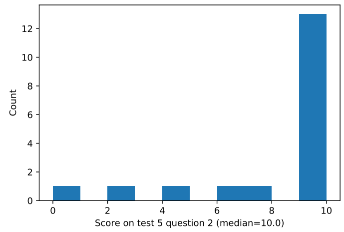
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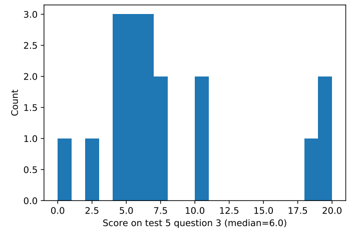
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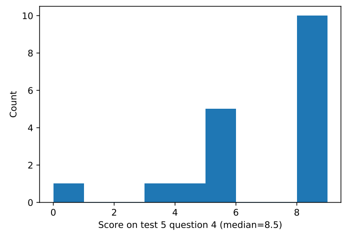
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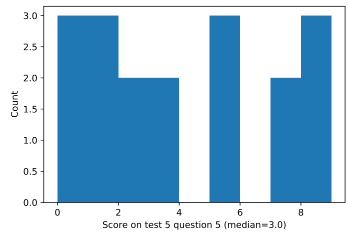
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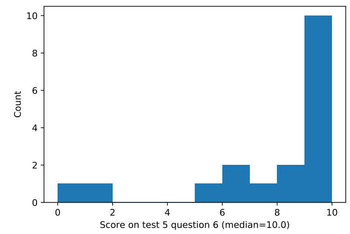
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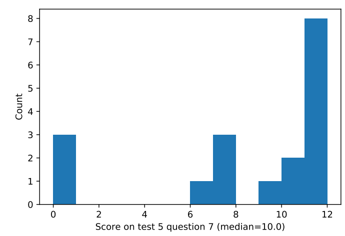
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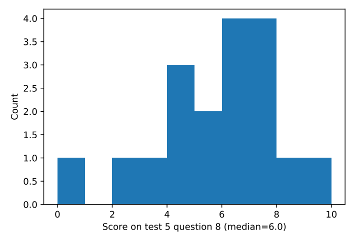
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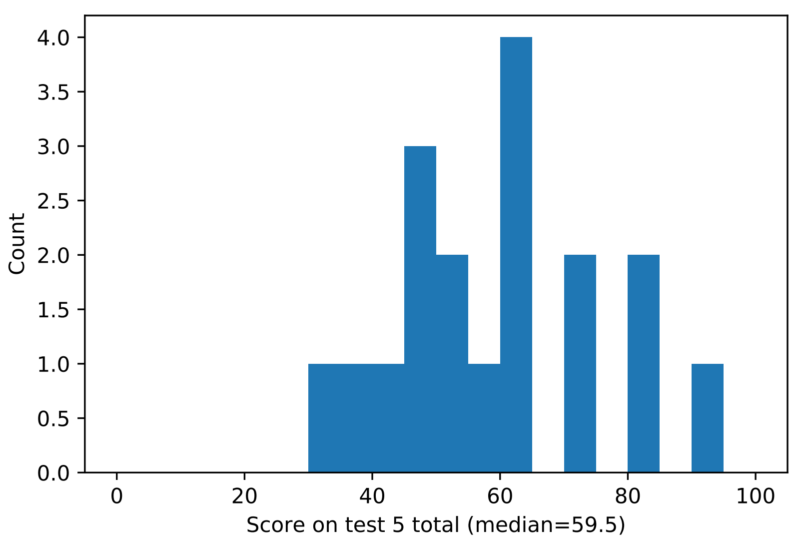
Score on test 5 question 8 (median=6.0)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Score on test 5 total (median=59.5)