score	possible	page
	20	1
	28	2
	32	3
	20	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Sketch the graph of a single function f that satisfies the following conditions.

/2 (a) f has domain $[0, 12)$.

/2 (b) $f(x) < 4$ always.

/2 (c) $f(1) = 2$

/2 (d) $\lim_{x\to 2^-} f(x) = 1$.

/2 (e) $\lim_{x\to 2^+} f(x) = 3$.

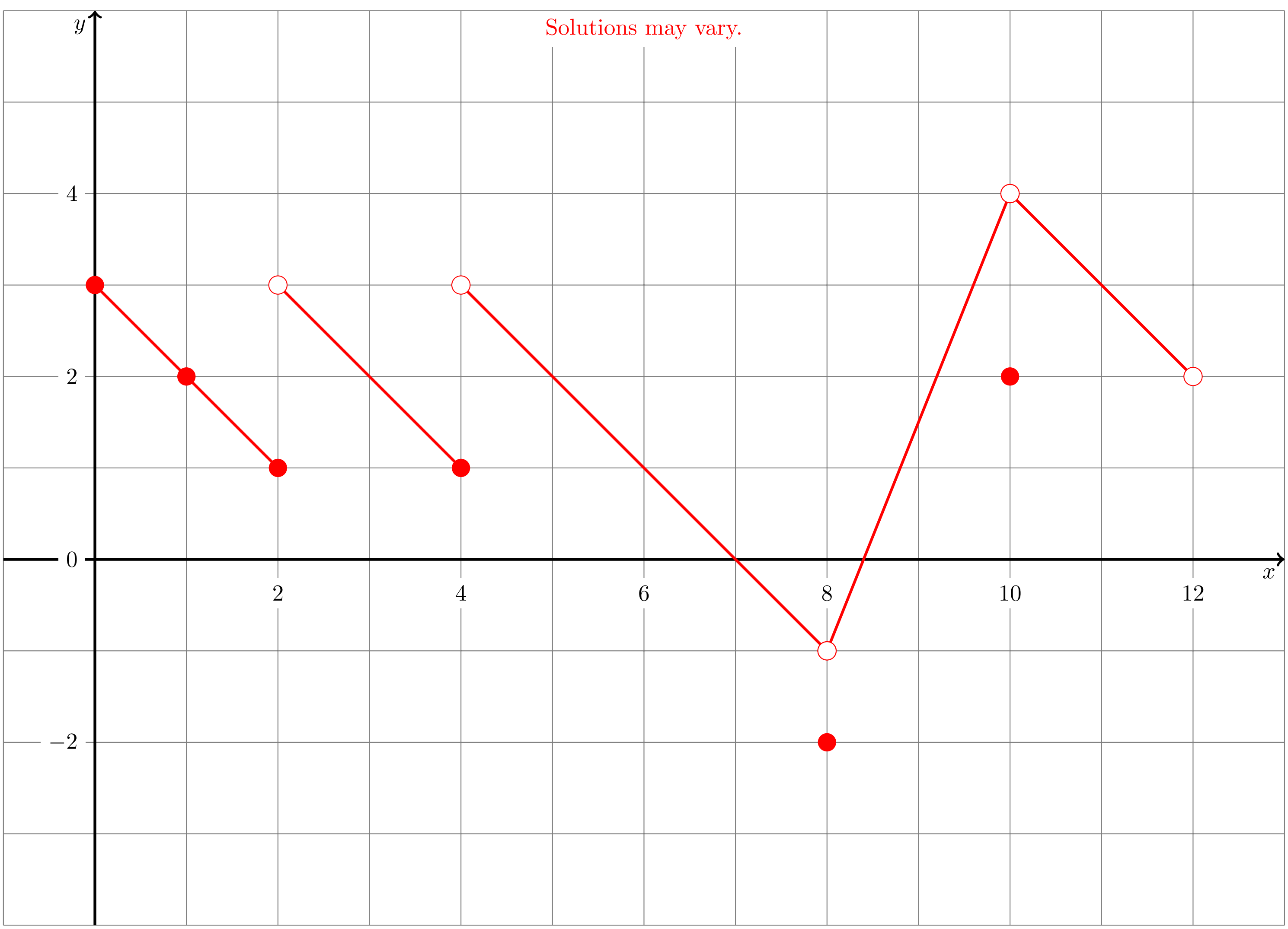
/2 (f) $\lim_{x\to 4} f(x)$ does not exist.

/2 (g) $\lim_{x\to 8} f(x) = -1$.

/2 (h) $f(8) = -2$

/2 (i) $\lim_{x\to 10} f(x) = 4$.

/2 (j) $\lim_{x\to 12^-} f(x) = 2$.



/5 2. (a) Complete the definition of *The Limit of a Function f at a point.*:
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The statement that "the limit of $f(x)$, as x approaches c, is L" is denoted by

$$\lim_{x\to c} f(x) = L,$$

and means that
given any $\varepsilon > 0$,
there exists $\delta > 0$ such that for all x in I, where $x \neq c$,
if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

/9 (b) Use this definition to prove that $\lim_{x\to 3} 5x + 4 = 19$.
Starting from $|(5x+4) - 19| = |f(x) - L| < \varepsilon$, we can simplify to $|5x - 15| < \varepsilon$ and then divide by 5 to get $|x - 3| < \varepsilon/5$. Given any $\varepsilon > 0$, we can choose $\delta = \varepsilon/5$ and have

$$|x-3| < \delta = \varepsilon/5 \Leftrightarrow |5x - 15| < \varepsilon \Leftrightarrow |(5x+4) - 19| = |f(x) - L| < \varepsilon\,,$$

so we have proven the limit.

/5 3. (a) Complete the statement of *The Squeeze Theorem*:
Let f, g and h be functions on an open interval I containing c such that for all x in I, (except possibly at $x = c$)

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

/9 (b) Use the Squeeze Theorem to evaluate $\lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right)$.
Set

$$\begin{aligned} f(x) &= -2|x|\,, \\ g(x) &= 2x\sin\left(\frac{1}{x}\right)\,, \quad \text{and} \\ h(x) &= 2|x|\,. \end{aligned}$$

Since $-1 \leq \sin(\cdot) \leq 1$, we have $f(x) \leq g(x) \leq h(x)$ when x is near 0 (and for all $x \neq 0$), so the first assumption of the Squeeze Theorem holds with $c = 0$.
We can compute $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so the second assumption of the Squeeze Theorem also holds with $c = 0$ and $L = 0$.
Thus the conclusion of the Squeeze Theorem holds and we can conclude

$$\lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) = \lim_{x\to c} g(x) = L = 0\,.$$

4. Compute the following limits. Use properties of limits and algebra, not the ε-δ definition. Do **not** use L'Hôpital's rule.

/8 (a) $\lim_{x\to 0} \frac{4^{-1}-(4-x)^{-1}}{x} =$

$$\lim_{x\to 0} \frac{\frac{1}{4}-\frac{1}{4-x}}{x} = \lim_{x\to 0} \frac{\frac{4-x-4}{4(4-x)}}{x} = \lim_{x\to 0} \frac{\frac{-x}{4(4-x)}}{x} = \lim_{x\to 0} \frac{-x}{x4(4-x)} = \lim_{x\to 0} \frac{-1}{4(4-x)} = \frac{-1}{4(4-0)} = \frac{-1}{16}$$

/8 (b) $\lim_{x\to 0} \frac{4^2-(4-x)^2}{x} =$

$$\lim_{x\to 0} \frac{4^2-(4^2-8x+x^2)}{x} = \lim_{x\to 0} \frac{8x-x^2}{x} = \lim_{x\to 0} \frac{x(8-x)}{x} = \lim_{x\to 0} 8-x = 8-0 = 8$$

/8 (c) $\lim_{x\to 0} \frac{x^2-7x}{x^2+2x} =$

$$\lim_{x\to 0} \frac{x(x-7)}{x(x+2)} = \lim_{x\to 0} \frac{(x-7)}{(x+2)} = \frac{(0-7)}{(0+2)} = -\frac{7}{2}$$

/8 (d) $\lim_{s\to 4} \frac{x-4}{\sqrt{x}-2} =$

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{x-4}{x-4}\frac{\sqrt{x}+2}{1} = \lim_{x\to 4} \sqrt{x}+2 = \sqrt{4}+2 = 2+2 = 4$$

5. Determine whether each of the following statements is True or False.
Correct answers are worth 2, incorrect answers are worth 0, and no answer is worth 1.

/2 (a) True False If $x \neq 0$, $y \neq 0$, and $z \neq 0$, then $\dfrac{\frac{x}{y}}{z} = \dfrac{x}{yz}$.
True. Dividing by y and then z is the same as dividing by yz.

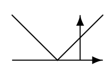
/2 (b) True False If $x \neq 0$ then $\left(\dfrac{25}{4x^4}\right)\left(\dfrac{5}{2x^3}\right)^{-3} = \dfrac{2x^5}{5}$.
True. $\left(\dfrac{25}{4x^4}\right)\left(\dfrac{5}{2x^3}\right)^{-3} = \left(\dfrac{25}{4x^4}\right)\left(\dfrac{2x^3}{5}\right)^{3} = \left(\dfrac{25}{4x^4}\right)\left(\dfrac{2^3x^9}{5^3}\right) = \dfrac{25 \cdot 2^3x^9}{4x^4 \cdot 5^3} = \dfrac{2x^5}{5}$.

/2 (c) True False If $x \neq 0$ then $x + x^{-1} = 0$.
False. If $x = 1$ then $x + x^{-1} = 2$.

/2 (d) True False $5 + \cos^2(\theta) + \sin^2(\theta) = 6$.
True. $\sin^2(\theta) + \cos^2(\theta) = 1$

/2 (e) True False If $x > 1$ then $7^{\log_7(x)} = x$.
True. Exponentials and logarithms are inverse functions. $x > 1$ is sufficient to stay in the domain of the logarithm.

/2 (f) True False If $x > 0$ then $\dfrac{\ln(x+2)}{\ln(x)} = \ln(2)$.
False. There are no simplifications for division of logarithms.

/2 (g) True False The graph of $f(x) = |x-1|$ looks like 
False. The argument $x - 1$ shifts the graph of $|x|$ to the right.

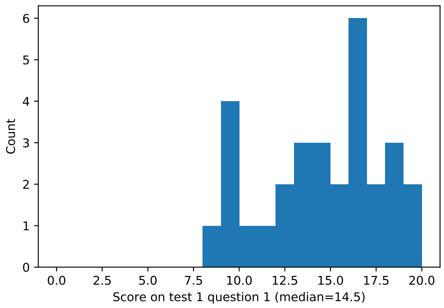
/2 (h) True False If $\lim\limits_{x\to 6} f(x) = 7$, then $\lim\limits_{t\to 3} f(2t) = 7$.
True. Informally, if t is near 3, then $2t$ is near 6 so $f(2t)$ is near 7. Formally, for any $\varepsilon > 0$, we know there exist $\delta_1 > 0$ such that $0 < |x-6| < \delta_1$ implies $|f(x) - 7| < \varepsilon$. Setting $2t = x$, we then know $0 < |2t-6| < \delta_1 \Leftrightarrow 0 < |t-3| < \delta_1/2 = \delta_2$ implies $|f(2t) - 7| < \varepsilon$.

/2 (i) True False $\lim\limits_{x\to 3^-} \dfrac{x-3}{|x-3|} = 1$.
False. Since $x \to 3^-$ we know $x < 3$ so $x - 3 < 0$ so $|x-3| = -(x-3)$. Thus

$$\lim_{x\to 3^-} \frac{x-3}{|x-3|} = \lim_{x\to 3^-} \frac{x-3}{-(x-3)} = \lim_{x\to 3^-} -1 = -1\,.$$

/2 (j) True False If $\lim\limits_{x\to 3} f(x) = 4$, then $\lim\limits_{x\to 3^+} f(x) = 4$.
True. For the ordinary limit to exist, both one-sided limits must exist and agree with it.

Scores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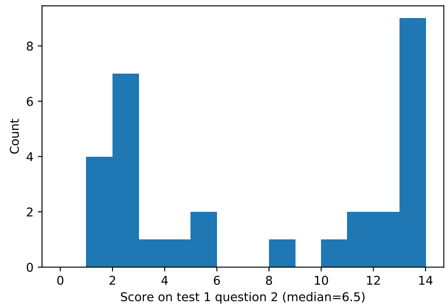
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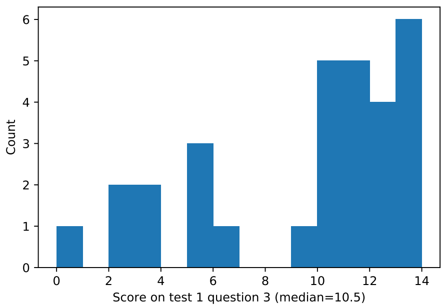
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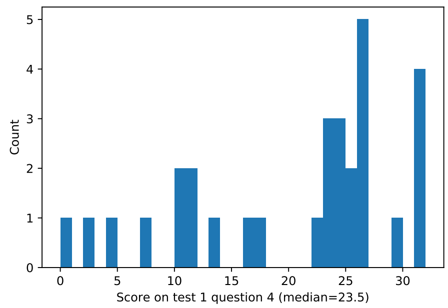
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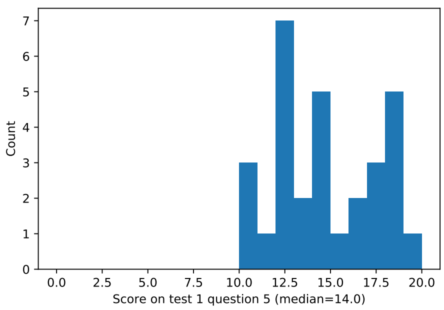
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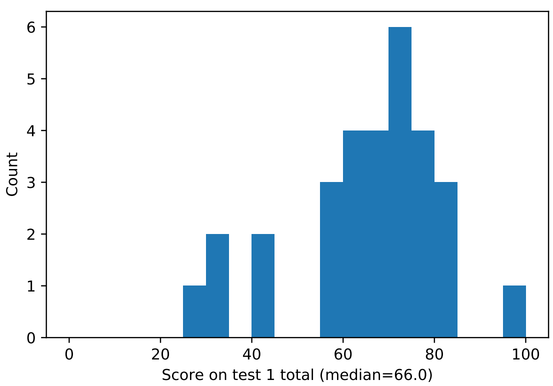
Count
Score on test 1 total (median=66.0)