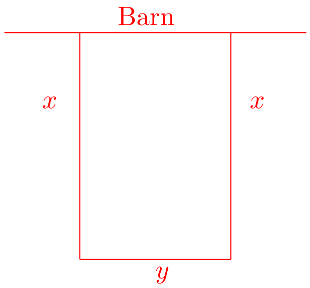
score	possible	page
	20	1
	25	2
	30	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/20 1. A farmer wants a rectangular pen next to a large barn, using the wall of the barn as one side of the pen. If the farmer wants the area enclosed to be $1800\,\text{m}^2$, what are the dimensions of the fence that minimize the length of fencing used? Make a sketch of the fence and barn, clearly showing the variables you are using.



We want to minimize $P = 2x + y$ and we have the constraint $xy = 1800\,\text{m}^2$.

Solving the constraint for y yields $y = x^{-1}1800\,\text{m}^2$. Substituting into P gives $P(x) = 2x + x^{-1}1800\,\text{m}^2$.

Differentiating gives $P'(x) = 2 - x^{-2}1800\,\text{m}^2$ and setting $P'(x) = 0$ gives

$$2 - x^{-2}1800\,\text{m}^2 = 0 \quad \Rightarrow \quad 2x^2 = 1800\,\text{m}^2 \quad \Rightarrow \quad x = \pm 30\,\text{m}$$

as the critical numbers. The domain is $0\,\text{m} < x < \infty\,\text{m}$, so we only use $x = 30\,\text{m}$.

Differentiating $P'(x)$ gives $P''(x) = 2x^{-3}1800\,\text{m}^2$, which is positive for $x > 0\,\text{m}$, so this critical number is a local minimum. (Alternatively: Testing the intervals $(0\,\text{m}, 30\,\text{m})$ and $(30\,\text{m}, \infty\,\text{m})$ yields $P'(1\,\text{m}) = 2 - 1800 < 0$ and $P'(100000\,\text{m}) = 2 - \text{tiny} > 0$, so there is a local minimum at $x = 30\,\text{m}$.)

Substituting back into the constraint gives $y = (30\,\text{m})^{-1}1800\,\text{m}^2 = 60\,\text{m}$. Thus the side parallel to the barn should be $60\,\text{m}$ and the sides perpendicular to the barn should be $30\,\text{m}$.

2. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −1, and no answer is worth +1.

/3 (a) True False If $f'(a)$ exists then $\lim_{t\to a} f(t) = f(a)$.
True. If $f'(a)$ exists then f is differentiable at a, which implies f is continuous at a, which by definition means $\lim_{t\to a} f(t) = f(a)$.

/3 (b) True False If a function has a horizontal asymptote, then the graph of the function cannot cross the horizontal asymptote.
False. The horizontal asymptote only tells us about $\lim_{x\to-\infty} f(x)$ or $\lim_{x\to\infty} f(x)$, not what happens in between. For example, $x/(1+x^2)$ has a horizontal asymptote $y = 0$ and crosses it at $x = 0$.

/3 (c) True False If f and g are increasing functions on an interval (a, b) then fg is an increasing function on (a, b).
False. Both $f(x) = x$ and $g(x) = x$ are increasing everywhere, but $f(x)g(x) = x^2$ is decreasing on $(-\infty, 0)$.

/3 (d) True False $\lim_{x\to0^+} \frac{\ln(x)}{x} = \infty$.
False. $\lim_{x\to0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty \neq \infty$.

/3 (e) True False $\frac{d}{dy}\big(\arctan(y^2)\big) = \frac{2y}{1+y^4}$.
True. Using the chain rule, $\frac{d}{dy}\big(\arctan(y^2)\big) = \frac{1}{1+(y^2)^2}(2y) = \frac{2y}{1+y^4}$.

/10 3. Let $f(x) = e^{-ax^2}$, where a is an unknown positive constant. Find the inflection points of f. Your answer may include a (and e).
Differentiating twice yields

$$f'(x) = e^{-ax^2}(-2ax) \quad \text{and}$$
$$f''(x) = e^{-ax^2}(-2ax)(-2ax) + e^{-ax^2}(-2a) = 2ae^{-ax^2}(2ax^2 - 1)\,.$$

Setting $f''(x) = 0$ and noting $2ae^{-ax^2} > 0$, we solve $2ax^2 - 1 = 0$ to get $x = \pm\sqrt{1/(2a)}$ as locations of possible inflection points. Using the sign chart

f	∪		∩		∪
f''	+	0	−	0	+
	$(-\infty, -\sqrt{1/(2a)})$	$-\sqrt{1/(2a)}$	$(-\sqrt{1/(2a)}, \sqrt{1/(2a)})$	$\sqrt{1/(2a)}$	$(\sqrt{1/(2a)}, \infty)$

we see that f does change concavity at $x = \pm\sqrt{1/(2a)}$, so these are the locations of inflection points. The actual points are

$$(-\sqrt{1/(2a)}, f(-\sqrt{1/(2a)})) = (-\sqrt{1/(2a)}, e^{-1/2}) \quad \text{and}$$
$$(\sqrt{1/(2a)}, f(\sqrt{1/(2a)})) = (-\sqrt{1/(2a)}, e^{-1/2})\,.$$

/10 4. Compute the following limit. Show work to justify your answer. If you use the Squeeze theorem or L'Hôpital's rule, then show that their assumptions are satisfied.

$\lim_{x\to 0} \frac{1-\cos(3x)}{x\sin(x)} =$

The numerator and denominator are both continuous (and differentiable) functions. Plugging in gives $\frac{1-\cos(0)}{0\sin(0)} = \frac{0}{0}$, which is indeterminate of the right form to use L'Hôpital's rule. Applying L'Hôpital's rule by differentiating the numerator and denominator gives

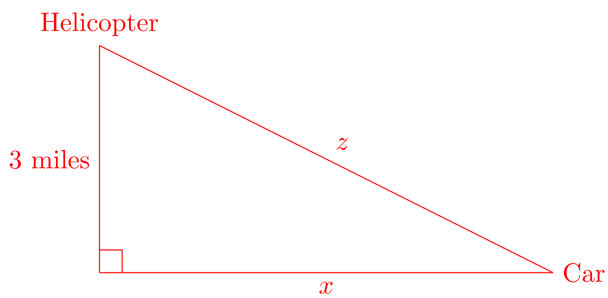
$$\lim_{x\to 0} \frac{\sin(3x)3}{\sin(x)+x\cos(x)}.$$

Plugging in gives $\frac{\sin(0)3}{\sin(0)+0\cos(0)} = \frac{0}{0}$, which is indeterminate of the right form to use L'Hôpital's rule. Applying L'Hôpital's rule by differentiating the numerator and denominator gives

$$\lim_{x\to 0} \frac{\cos(3x)3^2}{\cos(x)+\cos(x)-x\sin(x)}.$$

Plugging in gives $\frac{\cos(0)3^2}{\cos(0)+\cos(0)-0\sin(0)} = \frac{3^2}{1+1} = \frac{9}{2}$, which is thus the value of the original limit.

/20 5. A highway patrol helicopter is hovering 3 miles above a level, straight road. The pilot sees an oncoming car, and determines with radar that at the instant the line-of-sight distance from the helicopter to the car is 5 miles, the line-of-sight distance is decreasing at a rate of 80 miles per hour. Find the car's driving speed along the road. Include units in your answer.



We are told $\frac{dz}{dt}\big|_{z=5} = -80\,\frac{\text{mi}}{\text{hr}}$ and asked to find $-\frac{dx}{dt}\big|_{z=5}$.

We can relate x and z using the Pythagorean theorem to get $x^2+(3\,\text{mi})^2 = z^2$.

Differentiating gives $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$.

Solving for $\frac{dx}{dt}$ gives $\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$.

When $z = 5\,\text{mi}$ we have $x^2+(3\,\text{mi})^2 = (5\,\text{mi})^2$ so $x = 4\,\text{mi}$.

Thus

$$-\left.\frac{dx}{dt}\right|_{z=5} = -\left.\frac{z}{x}\right|_{z=5}\left.\frac{dz}{dt}\right|_{z=5} = -\frac{5\,\text{mi}}{4\,\text{mi}}\left(-80\frac{\text{mi}}{\text{hr}}\right) = 100\,\frac{\text{mi}}{\text{hr}}.$$

6. For the function $f(x) = \dfrac{x^2}{x^2-9}$, which has $f'(x) = \dfrac{-18x}{(x^2-9)^2}$ and $f''(x) = \dfrac{54(x^2+3)}{(x^2-9)^3}$:

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/4 (c) Find the intervals on which f is increasing or decreasing.

/4 (d) Find the local maximum and minimum values of f.

/6 (e) Find the intervals of concavity and the inflection points .

/5 (f) Use the information above to sketch the graph.

(f has even symmetry, so we could save half the work, but this is optional.)

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is 0 and there are vertical asymptotes at $x = -3$ and $x = 3$.

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-9} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = \lim_{x\to\pm\infty} \frac{1}{1} = 1$$

so there is a horizontal asymptote at $y = 1$.

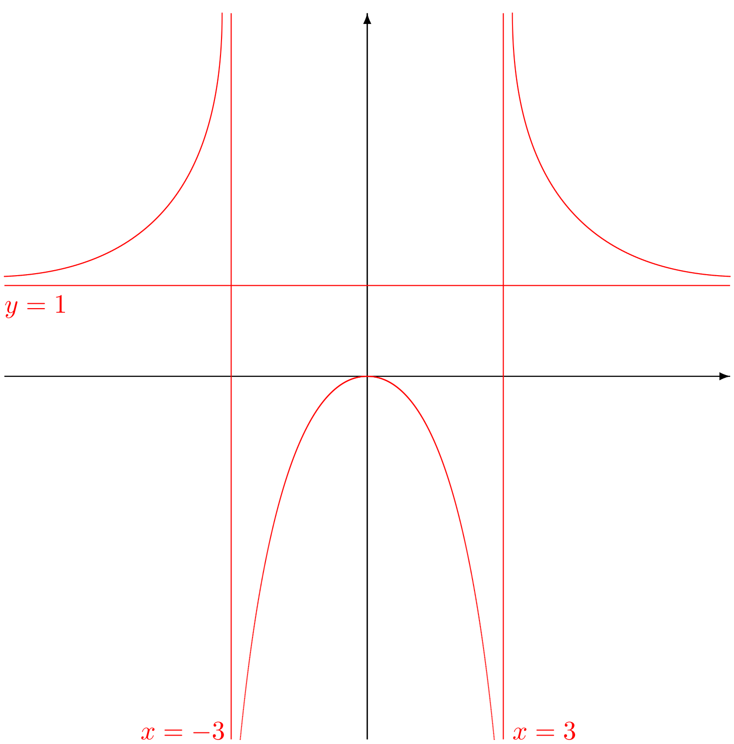
The given $f'(x) = \dfrac{-18x}{(x^2-9)^2}$ is zero at $x = 0$ and does not exist at the vertical asymptotes $x = -3$ and $x = 3$.

The given $f''(x) = \dfrac{54(x^2+3)}{(x^2-9)^3}$ is never zero and does not exist at the vertical asymptotes $x = -3$ and $x = 3$.

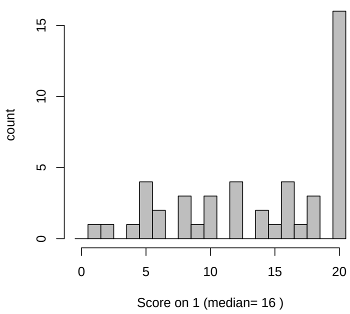
Assembling into a chart and checking signs, we have

f		V.A		$\to$		V.A.	
f''	$+$	DNE	$-$	$-$	$-$	DNE	$+$
f'	$+$	DNE	$+$	0	$-$	DNE	$-$
	$(-\infty, -3)$	-3	$(-3, 0)$	0	$(0, 3)$	3	$(3, \infty)$

There is a local maximum at $x = 0$ with value $f(0) = 0$ and no local minima. There are no inflection points.



Scores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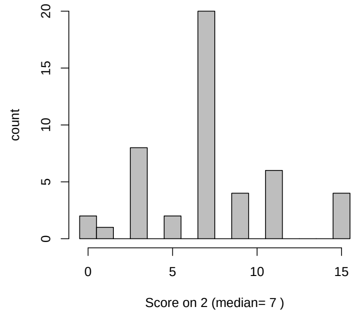
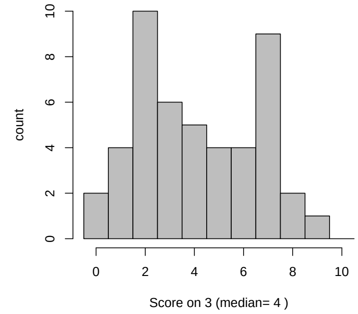
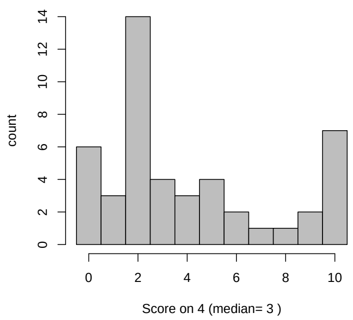
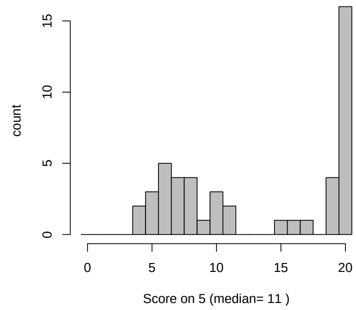
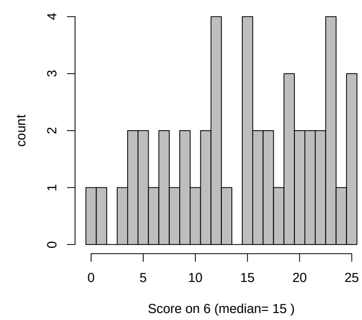
Score on 1 (median= 16)

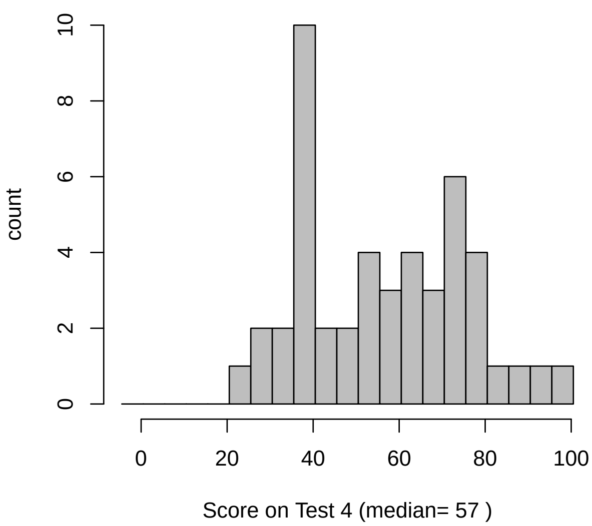

count
Score on 2 (median= 7)


count
Score on 3 (median= 4)


count
Score on 4 (median= 3)


count
Score on 5 (median= 11)


count
Score on 6 (median= 15)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Score on Test 4 (median= 57)