1. Determine whether each of the following statements is True or False. Correct answers are worth +3, incorrect answers are worth −1, and no answer is worth +1.

(a) True False If \( f(a) < N < f(b) \) then there exists \( c \in (a, b) \) such that \( f(c) = N \).
False. This is almost the Intermediate Value Theorem but lacks the assumption that \( f \) is continuous on \([a, b]\).

(b) True False \( \lim_{x \to 0^+} \frac{\ln(x)}{x} = -\infty \).
[3.7#9] True. \( \lim_{x \to 0^+} \frac{\ln(x)}{x} = -\infty \).

(c) True False A local maximum of a function \( f(x) \) can only occur at a point where \( f'(x) = 0 \).
False. It could also occur where \( f'(x) \) does not exist.

(d) True False If \( f \) is a continuous function on the interval \([a, b]\) then \( f \) attains an absolute maximum at some number \( c \) in \([a, b]\).
True. This is the Extreme Value Theorem in section 4.1.

(e) True False If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\) then \( f \) is constant on \((a, b)\).
True. This is Theorem 5 in section 4.2 and is a consequence of the Mean Value Theorem.

(f) True False If \( f'(x) > 2 \) for all \( x \) and \( f(0) = 5 \), then \( f(3) > 11 \).
[Similar to 4.3 #23, 25] True. The Mean Value Theorem implies there exists \( c \in (0, 3) \) with \( f'(c) = \frac{f(3)-5}{3}, \) so \( \frac{f(3)-5}{3} > 2 \), which implies \( f(3) > 11 \). (The simpler logic of saying \( f \) must be above the line through \((0, 5)\) with slope 2 also works.)

2. For the function \( f(x) = x^{1/3}(x + 4) \), find all \( c \) such that \( f''(c) = 0 \) or \( f''(c) \) does not exist.
[Similar to part of 4.3 #33] We can compute
\[
\begin{align*}
f'(x) &= (1/3)x^{-2/3}(x + 4) + x^{1/3} = (1/3)x^{-2/3}((x + 4) + 3x) = (4/3)x^{-2/3}(x + 1) \quad \text{and} \\
f''(x) &= (4/3) \left((-2/3)x^{-5/3}(x + 1) + x^{-2/3}\right) = (4/9)x^{-5/3}((-2)(x + 1) + 3x) = (4/9)x^{-5/3}(x - 2).
\end{align*}
\]
\( f''(c) = 0 \) when \( c = 2 \) and \( f''(c) \) DNE when \( c = 0 \).
3. Compute the limits. Show your work and/or explain your reasoning.

/5 (a) \( \lim_{x \to \infty} x^2 e^{-x^2} = \)

[Similar to 3.7 #25] Plugging in yields \( \infty \cdot 0 \), which is indeterminate but not in the form for L’Hôpital’s rule. Rewriting to get \( \infty/\infty \) form, we can apply L’Hôpital’s rule to get

\[
\lim_{x \to \infty} x^2 e^{-x^2} = \lim_{x \to \infty} \frac{2x}{2xe^{x^2}} = \lim_{x \to \infty} \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0.
\]

/5 (b) \( \lim_{x \to 0} 3x^2 \sin \left( \frac{1}{x} \right) = \)

[Similar to 1.4 Example 9] Set

\[
f(x) = -3x^2, \\
g(x) = 3x^2 \sin \left( \frac{1}{x} \right), \text{ and} \\
h(x) = 3x^2.
\]

Since \( |\sin(\cdot)| \leq 1 \), we have \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near 0 (and for all \( x \neq 0 \)). We can compute \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \), so the assumptions of the Squeeze Theorem are satisfied and we can conclude \( \lim_{x \to 0} g(x) = 0 \).

/5 4. (a) State the Mean Value Theorem (MVT) using the template below.

If

- \( f \) is continuous on the closed interval \([a, b]\) and
- \( f \) is differentiable on the open interval \((a, b)\),

then there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

/5 (b) Verify that the function \( f(x) = 1/x \) satisfies each of the hypotheses of the MVT on the interval \([1, 3]\). Then find all numbers \( c \) that satisfy the conclusion of the MVT.

[4.2 #12] \( f(x) = 1/x \) is differentiable on \((0, \infty)\) and hence differentiable on \((1, 3)\) and continuous on \([1, 3]\).

We have \( f'(x) = -x^{-2} \) and \( \frac{f(3) - f(1)}{3 - 1} = \frac{-1 - 1}{2} = -1 \). Setting \( -x^{-2} = -\frac{1}{3} \) and solving gives \( x = \pm \sqrt{3} \), of which \( \sqrt{3} \in (1, 3) \).
5. Find the absolute maximum and minimum values of \( f(x) = x^3 - 9x^2 + 1 \) on the interval \([-2, 2]\).

[Similar to 4.1 #40] \( f'(x) = 3x^2 - 18x = 3x(x - 6) \) so the critical numbers are \( x = 0 \) and \( x = 6 \). Since \( x = 6 \) is not in the interval, we discard it. Evaluating at \( x = 0 \) and at the endpoints we get

\[
\begin{align*}
f(0) &= 1, \\
f(-2) &= (-2)^3 - 9(-2)^2 + 1 = -8 - 36 + 1 = -43, \quad \text{and} \\
f(2) &= 2^3 - 9 \cdot 2^2 + 1 = 8 - 36 + 1 = -27.
\end{align*}
\]

Thus the absolute maximum is 1 and occurs at \( x = 0 \) and the absolute minimum is \(-43\) and occurs at \( x = -2 \).

6. Sketch the graph of a single function \( f \) that has all of the following properties:

- \( f'(1) = f'(-1) = 0 \).
- \( f'(x) < 0 \) if \( |x| < 1 \).
- \( f'(x) > 0 \) if \( 1 < |x| < 2 \).
- \( f'(x) = -1 \) if \( |x| > 2 \).
- \( f''(x) < 0 \) if \( -2 < x < 0 \).
- An inflection point at \( (0, 1) \).

[4.3 #20] Since \( f'(x) = -1 \) if \( |x| > 2 \), we know \( f''(x) = 0 \) if \( |x| > 2 \). Since there is an inflection point at \( (0, 1) \) and \( f''(x) < 0 \) for \(-2 < x < 0\), we know \( f''(x) > 0 \) for \( 0 < x < 2 \). Organizing into a chart, we have

\[
\begin{array}{cccccccc}
  \quad & f'' & \searrow & (\rightarrow) & \quad & \nearrow & \downarrow & \quad IP \quad & \leftarrow & \quad \nearrow & \downarrow \\
  f' & -1 & + & 0 & - & 0 & + & 0 \\
  \hline
  (-\infty, -2) & -2 & (-2, -1) & -1 & (-1, 0) & 0 & (0, 1) & 1 & (1, 2) & 2 & (2, \infty)
\end{array}
\]

It is consistent if we make \( f''(x) > 0 \) if \( 0 < x < 2 \). At \(-2 \) and \( 2 \) there must be a cusp or discontinuity.

Here is one solution:
7. For the function \( f(x) = \frac{x}{x^2 + 4} \), which has \( f'(x) = \frac{(2 + x)(2 - x)}{(x^2 + 4)^2} \) and \( f''(x) = \frac{2x(2 + \sqrt{3})(x - 2\sqrt{3})}{(x^2 + 4)^3} \):

(a) Find the \( x\)- and \( y\)-intercepts.

(b) Find any asymptotes.

(c) Find the intervals on which \( f \) is increasing or decreasing.

(d) Find the local maximum and minimum values of \( f \).

(e) Find the intervals of concavity and the inflection points.

(f) Use the information above to sketch the graph.

\( f \) has odd symmetry, so we could save half the work, but this is optional.)

\( f(0) = 0 \) and no other \( x \) makes \( f(x) = 0 \), so both intercepts are at \((0, 0)\).

The denominator is never 0 so there are no vertical asymptotes.

\[
\lim_{x\to\pm\infty} \frac{x}{x^2 + 4} = \lim_{x\to\pm\infty} \frac{1}{x} = 0
\]
so there is a horizontal asymptote at \( y = 0 \).

The given \( f'(x) = \frac{(2 + x)(2 - x)}{(x^2 + 4)^2} \) is zero at \( x = -2 \) and \( x = 2 \).

The given \( f''(x) = \frac{2x(2 + \sqrt{3})(x - 2\sqrt{3})}{(x^2 + 4)^3} \) is 0 at \( x = 0 \), \( x = -2\sqrt{3} \), and \( x = 2\sqrt{3} \).

Assembling into a chart and checking signs, we have

\[
\begin{array}{cccccc}
\text{I.P.} & \overbrace{\curvearrowright} & \overbrace{\curvearrowleft} & \text{I.P.} & \overbrace{\curvearrowright} & \overbrace{\curvearrowleft} & \text{I.P.} & \overbrace{\curvearrowright} & \overbrace{\curvearrowleft} & \overbrace{\curvearrowleft} & \overbrace{\curvearrowright} & \overbrace{\curvearrowleft} \\
& \frac{(-\infty, -2\sqrt{3})}{-\sqrt{3}} & & \frac{(-2\sqrt{3}, -2)}{-2} & & \frac{(-2, 0)}{0} & & \frac{(0, 2)}{2} & & \frac{(2\sqrt{3})}{2\sqrt{3}} & & \frac{(2\sqrt{3}, \infty)}{-1} \\
& \frac{f'}{-} & & \frac{f''}{-} & & \frac{0}{+} & & \frac{0}{+} & & \frac{0}{0} & & \frac{0}{-} & & \frac{0}{-} \\
& \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}} & & \frac{y = 0 \text{ asymptote}}{y = 0 \text{ asymptote}}
\end{array}
\]

There is a local max at \( x = 2 \) with value \( f(2) = 2/(2^2 + 4) = 1/4 \) and a local min at \( x = -2 \) with value \( f(-2) = -2/(2^2 + 4) = -1/4 \). There are inflection points at \((-2\sqrt{3}, f(-2\sqrt{3})) = (-2\sqrt{3}, -2\sqrt{3}/(12 + 4)) = (-2\sqrt{3}, -\sqrt{3}/8)\), \((0, 0)\), and \((2\sqrt{3}, f(2\sqrt{3})) = (2\sqrt{3}, 3\sqrt{3}/8)\).
Scores

Score on 1 (median= 8 )

Score on 2 (median= 4 )

Score on 3a (median= 2 )

Score on 3b (median= 1 )

Score on 4a (median= 4 )

Score on 4b (median= 3 )

Score on 5 (median= 9 )

Score on 6 (median= 13 )

Score on 7 (median= 16 )

Score on Test 5 (median= 55 )