score	possible	page
	30	1
	20	2
	25	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.
Assume that the orders of the matrices are compatible so that they can be added or multiplied.

/3 (a) True False If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A}^2$ is similar to $\mathbf{B}^2$.
True. Similar means $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ for some invertible $\mathbf{B}$ so $\mathbf{B}^2 = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}$, which means $\mathbf{A}^2$ is similar to $\mathbf{B}^2$.

/3 (b) True False $\left(\mathbf{D}\left(\left(\mathbf{A}\mathbf{B}^T\right)^{-1}\mathbf{C}\right)^T\right)^{-1} = \mathbf{B}\mathbf{A}^T\mathbf{C}^{-T}\mathbf{D}^{-1}$

True, $\left(\mathbf{D}\left(\left(\mathbf{A}\mathbf{B}^T\right)^{-1}\mathbf{C}\right)^T\right)^{-1} = \left(\mathbf{D}\left(\mathbf{B}^{-T}\mathbf{A}^{-1}\mathbf{C}\right)^T\right)^{-1} = \left(\mathbf{D}\mathbf{C}^T\mathbf{A}^{-T}\mathbf{B}^{-1}\right)^{-1} = \mathbf{B}\mathbf{A}^T\mathbf{C}^{-T}\mathbf{D}^{-1}$.

/3 (c) True False $\operatorname{trace}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 4$.
False. Trace is the sum of diagonal elements so for this matrix it is $1 + 4 = 5$.

/3 (d) True False If $\mathbf{A}$ is invertible, then $\left((-1/2)\mathbf{A}^{-1}\right)^T$ is invertible.
True, $\left(\left((-1/2)\mathbf{A}^{-1}\right)^T\right)^{-1} = -2\mathbf{A}^T$.

/3 (e) True False If $\mathbf{A}$ is diagonalizable, then it is also invertible.
False. A matrix that is all 0 is diagonal already but is not invertible.

/3 (f) True False If λ is an eigenvalue of $\mathbf{B}$ and $\mathbf{B}$ is invertible, then λ is an eigenvalue of $\mathbf{B}^{-1}$.
False, $1/\lambda$ is an eigenvalue of $\mathbf{B}^{-1}$ but λ need not be.

/3 (g) True False If $\mathbf{A}$ is invertible, then it is also similar to a matrix in Jordan normal form.
True. Every (square) matrix is similar to a matrix in Jordan normal form. The assumption of invertible is not needed, except that it tells us the matrix is square.

/3 (h) True False $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.
False. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$, which is not a multiple of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

/3 (i) True False If the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$ is *linearly dependent*, then the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ is *linearly dependent*.
False, since if you start with a set $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ that is linearly independent, you can add $\mathbf{V}_4 = \mathbf{V}_3$ to it and get $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$, which is linearly dependent since $0 = \mathbf{V}_3 - \mathbf{V}_4$.

/3 (j) True False $\det(\mathbf{A})$ is equal to the sum of the eigenvalues of $\mathbf{A}$ (counted with their algebraic multiplicities).
False. $\det(\mathbf{A})$ is equal to the *product* of the eigenvalues of $\mathbf{A}$ (counted with their algebraic multiplicities).

2. For each matrix below, find all the eigenvalues.

/5 (a) $\begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 3 \\ -1 & 6-\lambda \end{vmatrix}$$
$$= (2-\lambda)(6-\lambda) - 3(-1) = 12 - 8\lambda + \lambda^2 + 3 = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3)\,,$$

so the eigenvalues are $\{3, 5\}$.

/5 (b) $\begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix}$$
$$= (-1-\lambda)(-3-\lambda) - (-1)1 = 3 + 4\lambda + \lambda^2 + 1 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2\,,$$

so the only eigenvalue is -2.

/5 (c) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 3 \\ -1 & 0 & 3 \end{bmatrix}$

Expanding down the second column, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 2 & 3-\lambda & 3 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix}$$
$$= (3-\lambda)((3-\lambda)^2 - (-1)^2) = (3-\lambda)(9-6\lambda+\lambda^2-1) = (3-\lambda)(\lambda^2-6\lambda+8) = (3-\lambda)(\lambda-4)(\lambda-2)\,,$$

so we have eigenvalues $\{2, 3, 4\}$.

/5 (d) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ -7 & 9 & 0 & 0 \\ -16 & 17 & 4 & 0 \\ 0 & 0 & 1 & 11 \end{bmatrix}$

Since the matrix is triangular, its diagonal entries are its eigenvalues, so we have $\{3, 9, 4, 11\}$.

3. For each matrix below, the eigenvalues are given. For each eigenvalue, find a basis (of eigenvectors) for the eigenspace.

/10 (a) $\begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix}$ has eigenvalues 2 and 4.

For each eigenvalue, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. For $\lambda = 2$ this yields

$$\left[\begin{array}{cc|c} 6 & -6 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so x_2 is free and $x_1 = x_2$. Choosing $x_2 = 1$ gives an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 4$ this yields

$$\left[\begin{array}{cc|c} 4 & -6 & 0 \\ 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so x_2 is free and $x_1 = (3/2)x_2$. Choosing $x_2 = 2$ gives an eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

/15 (b) $\begin{bmatrix} 4 & -2 & -1 \\ 2 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}$ has eigenvalues 1 and 2.

For each eigenvalue, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. For $\lambda = 1$, using row operations $R_2 \mapsto R_2 - (2/3)R_1$, $R_3 \mapsto R_3 - (2/3)R_1$, and $R_3 \mapsto R_3 + 2R_2$, this yields

$$\left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ 0 & 1/3 & -1/3 & 0 \\ 0 & -2/3 & 2/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting x_3 be free, we have $x_2 = x_3$ and $x_1 = (2x_2 + x_3)/3 = x_3$. Choosing $x_3 = 1$ gives an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, using row operations $R_2 \mapsto R_2 - R_1$ and $R_3 \mapsto R_3 - R_1$, this yields

$$\left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 2 & -2 & -1 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting x_2 and x_3 be free, we have $x_1 = (2x_2 + x_3)/2$ and general solution

$$\mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix},$$

so there is a basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

/10 4. The matrix $\mathbf{A}$ has eigenvalues

- $\lambda_1 = 3$ with algebraic multiplicity 1 whose eigenspace has a basis $\left\{ \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T \right\}$ and
- $\lambda_2 = 5$ with algebraic multiplicity 2 whose eigenspace has a basis $\left\{ \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}^T \right\}$.

Find a matrix $\mathbf{P}$ that is invertible and a matrix $\mathbf{D}$ that is diagonal such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Placing the eigenvalues on the diagonal yields

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Placing the corresponding eigenvectors as columns in $\mathbf{P}$ yields

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}.$$

/15 5. The matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 9 \\ 0 & -16 & 17 \end{bmatrix}$ has eigenvalues

- $\lambda_1 = 3$ with algebraic multiplicity 1 whose eigenspace has a basis $\left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \right\}$ and
- $\lambda_2 = 5$ with algebraic multiplicity 2 whose eigenspace has a basis $\left\{ \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^T \right\}$.

Find a matrix $\mathbf{P}$ that is invertible and a matrix $\mathbf{J}$ that is in Jordan normal form such that $\mathbf{A} = \mathbf{PJP}^{-1}$.

Since $\lambda_1 = 3$ has algebraic and geometric multiplicity 1, it gives a Jordan block $\mathbf{J}_1 = \begin{bmatrix} 3 \end{bmatrix}$. Since $\lambda_2 = 5$ algebraic multiplicity 2 and geometric multiplicity 1, it gives a Jordan block $\mathbf{J}_2 = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. Assembling these gives

$$\mathbf{J} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

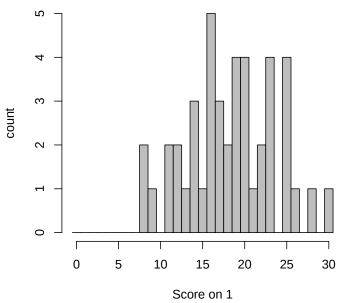
To replace the missing linearly independent eigenvector for λ_2, we solve $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^T$. Using the row-reduction $R_3 \mapsto R_3 - (4/3)R_2$ we obtain

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -12 & 9 & 3 \\ 0 & -16 & 12 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -12 & 9 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which means $x_1 = 0$, x_3 is free, and $x_2 = (3 - 9x_3)/(-12)$. Choosing $x_3 = 0$ gives the vector $\begin{bmatrix} 0 & -1/4 & 0 \end{bmatrix}^T$. Assembling the linearly independent vectors in the order used for $\mathbf{J}$ yields

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1/4 \\ 0 & 4 & 0 \end{bmatrix}.$$

Scores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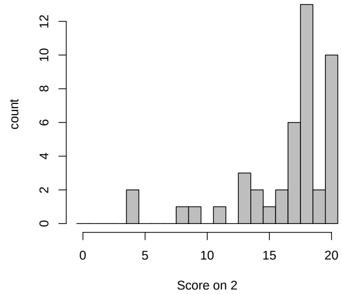
Score on 1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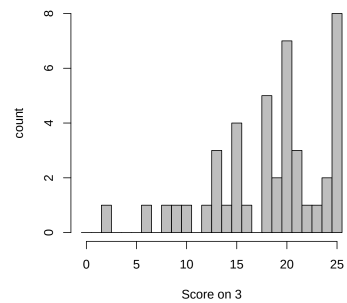
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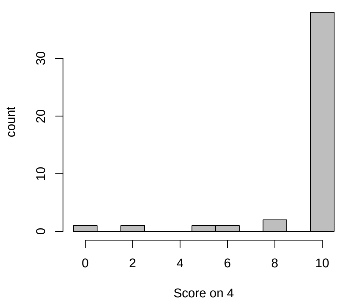
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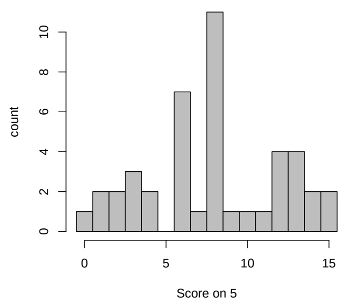
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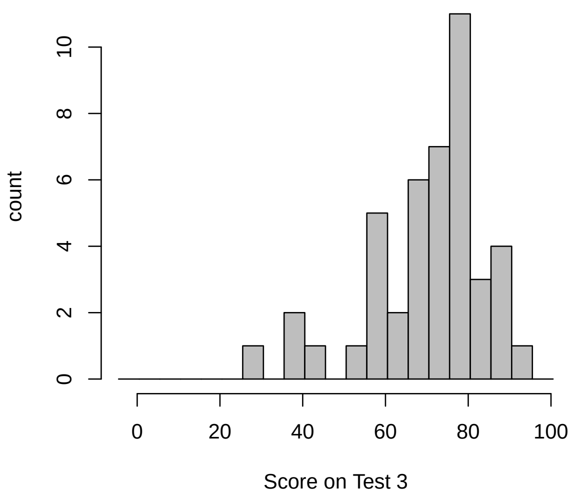
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Score on Test 3