score	possible	page
	30	1
	25	2
	25	3
	20	4
	100	

Name: ____________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.
Assume that the orders of the matrices are compatible so that they can be added or multiplied.

/3 (a) True False If $\mathbf{A}$ is any matrix and $\mathbf{B}$ is a row-reduced matrix obtained from $\mathbf{A}$ by performing a sequence of elementary row operations, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$.
True, this is how we determine the number of linearly independent rows in $\mathbf{A}$, which is our definition of rank. (This is also Section 2.6 Theorem 2.)

/3 (b) True False If $\mathbf{A}$ is symmetric and invertible, then $\mathbf{A} + \mathbf{A}^{-1}\mathbf{A}$ is symmetric.
True, $\left(\mathbf{A} + \mathbf{A}^{-1}\mathbf{A}\right)^T = (\mathbf{A} + \mathbf{I})^T = \mathbf{A}^T + \mathbf{I}^T = \mathbf{A} + \mathbf{I}$.

/3 (c) True False If $\mathbf{A}$ is symmetric and invertible, then $\mathbf{A} + \mathbf{A}^{-1}\mathbf{A}$ is invertible.
False, for example if $\mathbf{A} = -\mathbf{I}$ then $\mathbf{A} + \mathbf{A}^{-1}\mathbf{A} = -\mathbf{I} + \mathbf{I} = 0$.

/3 (d) True False If $\mathbf{L}$ is lower-triangular and invertible then $\mathbf{L}^{-1}$ is lower-triangular.
True, since our method of constructing the inverse will only create non-zero entries in the lower-triangular part. (This is Section 3.4 Property 7.)

/3 (e) True False $\left(\mathbf{D}\left(\left(\mathbf{A}\mathbf{B}^T\right)^{-1}\mathbf{C}\right)^T\right)^{-1} = \mathbf{B}\mathbf{A}^T\mathbf{C}^{-T}\mathbf{D}^{-1}$

True, $\left(\mathbf{D}\left(\left(\mathbf{A}\mathbf{B}^T\right)^{-1}\mathbf{C}\right)^T\right)^{-1} = \left(\mathbf{D}\left(\mathbf{B}^{-T}\mathbf{A}^{-1}\mathbf{C}\right)^T\right)^{-1} = \left(\mathbf{D}\mathbf{C}^T\mathbf{A}^{-T}\mathbf{B}^{-1}\right)^{-1} = \mathbf{B}\mathbf{A}^T\mathbf{C}^{-T}\mathbf{D}^{-1}$.

/3 (f) True False If the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$ is *linearly dependent*, then the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ is *linearly dependent*.
False, since if you start with a set $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ that is linearly independent, you can add $\mathbf{V}_4 = \mathbf{V}_3$ to it and get $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$, which is linearly dependent since $0 = \mathbf{V}_3 - \mathbf{V}_4$.

/3 (g) True False If the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ is a *basis*, then it is *linearly dependent*.
False, a *basis* (for some other set of vectors) is a *spanning set* that is also *linearly independent*.

/3 (h) True False $\mathbf{A}\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\mathbf{A}$.
True, both equal $\mathbf{I}$.

/3 (i) True False If all the entries in $\mathbf{A}$ are positive, then the determinant of $\mathbf{A}$ is positive.
False, for example $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$.

/3 (j) True False If $\mathbf{A}$ is invertible, then $\left((-1/2)\mathbf{A}^{-1}\right)^T$ is invertible.
True, $\left(\left((-1/2)\mathbf{A}^{-1}\right)^T\right)^{-1} = -2\mathbf{A}^T$.

2. Given $\mathbf{A}$ and $\mathbf{b}$, one method to solve $\mathbf{Ax} = \mathbf{b}$ starts by forming the augmented matrix $\mathbf{A}^{\mathbf{b}}$ and then using elementary row operations to transform it to row-reduced form. In each part below, the row-reduced form of $\mathbf{A}^{\mathbf{b}}$ is given. Determine whether or not the system is consistent and find all solutions.

/5 (a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Since $\text{rank}(\mathbf{A}) = 2 < \text{rank}(\mathbf{A}^{\mathbf{b}}) = 3$, the system is inconsistent and so has no solutions.

/5 (b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Since $\text{rank}(\mathbf{A}) = 2 = \text{rank}(\mathbf{A}^{\mathbf{b}})$, the system is consistent. Since $\mathbf{A}$ has 3 columns there are 3 variables and so $3 - \text{rank}(\mathbf{A}) = 1$ arbitrary parameter. Letting $t = x_3$ be this parameter, we have $x_2 = 6 - 5t$ and $x_1 = 4 - 2(6 - 5t) - 3t = -8 + 7t$. In vector form this is $\mathbf{x} = \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$.

We can check

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -8 + 12 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 - 10 + 3 \\ -5 + 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}.$$

/5 (c) $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Since $\text{rank}(\mathbf{A}) = 2 < \text{rank}(\mathbf{A}^{\mathbf{b}}) = 3$, the system is inconsistent and so has no solutions.

/5 (d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Since $\text{rank}(\mathbf{A}) = 1 = \text{rank}(\mathbf{A}^{\mathbf{b}})$, the system is consistent. Since $\mathbf{A}$ has 3 columns there are 3 variables and so $3 - \text{rank}(\mathbf{A}) = 2$ arbitrary parameters. Letting $t = x_3$ and $s = x_2$ be these parameters, we have $x_1 = 4 - 2s - 3t$. In vector form this is $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

We can check

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 + 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 + 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

/5 (e) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right]$

Since $\text{rank}(\mathbf{A}) = 3 \text{rank}(\mathbf{A}^{\mathbf{b}}) = 3$, the system is consistent and has a unique solution. Using back-substitution, $x_3 = 7$, so $x_2 = 6 - 5(7) = -29$, so $x_1 = 4 - 2(-29) - 3(7) = 4 + 58 - 21 = 41$. We can check

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 41 \\ -29 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 + 2(-29) + 3(7) \\ -29 + 5(7) \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}.$$

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 4 & 6 & 0 \end{bmatrix}$.

/15 (a) Compute $\mathbf{A}^{-1}$.

Augmenting $\mathbf{A}$ with $\mathbf{I}$ and then applying $R_3 \mapsto R_3 - 4R_1$ and then $R_3 \mapsto R_3 + 2R_2$ yields

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -4 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & 2 & 1 \end{array}\right].$$

Applying $R_3 \mapsto (1/2)R_3$, then $R_2 \mapsto R_2 - R_3$, and then $R_1 \mapsto R_1 - 2R_2$, yields

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1/2 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 1/2 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 1/2 \end{array}\right],$$

so $\mathbf{A}^{-1} = \begin{bmatrix} -3 & 0 & 1 \\ 2 & 0 & -1/2 \\ -2 & 1 & 1/2 \end{bmatrix}$.

/5 (b) Multiply $\mathbf{A}\mathbf{A}^{-1}$ to check that it equals $\mathbf{I}$. (Show your work.)

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & 1 \\ 2 & 0 & -1/2 \\ -2 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -3+2(2) & 0 & 1(1)+2(-1/2) \\ 1(2)+1(-2) & 1 & 1(-1/2)+1(1/2) \\ 4(-3)+6(2) & 0 & 4(1)+6(-1/2) \end{bmatrix} = \mathbf{I}.$$

/10 (c) Compute the $\mathbf{LU}$ decomposition of $\mathbf{A}$.

Applying $R_3 \mapsto R_3 - 4R_1$ and then $R_3 \mapsto R_3 + 2R_2$ yields

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 4 & 6 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{U}.$$

Parsing the steps we took yields

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}.$$

4. Compute the following determinants.

/5 (a) $\begin{vmatrix} a & 2-a \\ 5 & \pi \end{vmatrix} =$

$a\pi - (2-a)5$

/15 (b) $\begin{vmatrix} 2 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 3 & 7 & 0 & -4 \\ 0 & -2 & 4 & 0 \end{vmatrix} =$

Expanding down the first column gives

$$2(-1)^{1+1}\begin{vmatrix} 0 & 1 & -1 \\ 7 & 0 & -4 \\ -2 & 4 & 0 \end{vmatrix} + 0 + 3(-1)^{3+1}\begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & -1 \\ -2 & 4 & 0 \end{vmatrix} + 0\,.$$

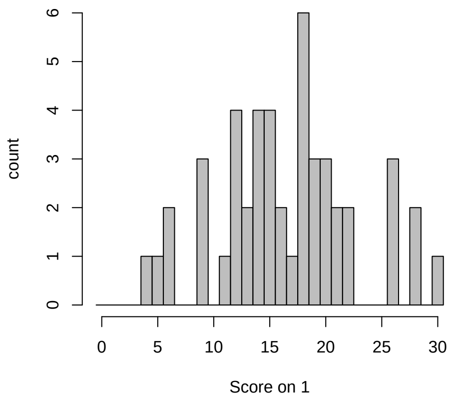
Simplifying and then expanding the 3×3 determinants along the first row gives

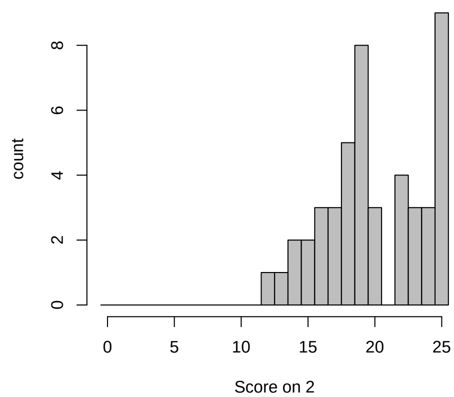
$$2\left(0 + 1(-1)^{1+2}\begin{vmatrix} 7 & -4 \\ -2 & 0 \end{vmatrix} + (-1)(-1)^{1+3}\begin{vmatrix} 7 & 0 \\ -2 & 4 \end{vmatrix}\right) + 3\left(5(-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} + 0 + 0\right)\,.$$

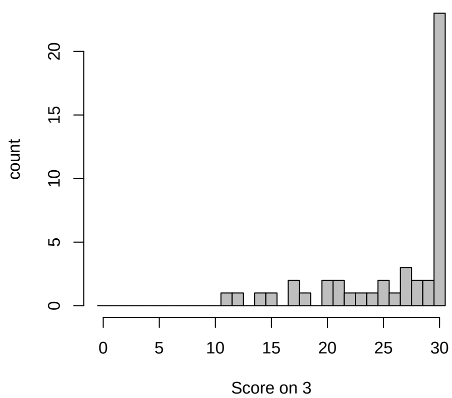
Simplifying and then evaluating the 2×2 determinants gives

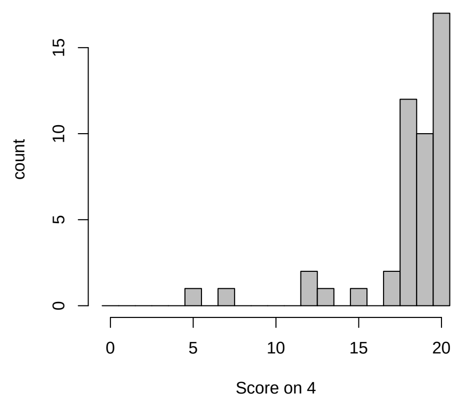
$$2\left(-(0-(-4)(-2)) - (7(4)+0)\right) + 3\left(5(0-(-1)4)\right) = 2(8-28) + 60 = -40 + 60 = 20\,.$$

Scores









There were accidentally 105 points on the test.