score	possible	page
	30	1
	20	2
	25	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.
Assume that the orders of the matrices are compatible so that they can be added or multiplied.

/3 (a) True False $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$.
True, since row rank equals column rank.

/3 (b) True False $\left(\mathbf{A} + \mathbf{A}^T\right)^T = \mathbf{A} + \mathbf{A}^T$.
True, $\left(\mathbf{A} + \mathbf{A}^T\right)^T = \mathbf{A}^T + \mathbf{A} = \mathbf{A} + \mathbf{A}^T$.

/3 (c) True False If $\mathbf{L}$ is lower-triangular then $\mathbf{L}^2$ is lower-triangular .
True. The product of lower-triangular matrices is always lower-triangular.

/3 (d) True False If $\mathbf{L}$ is lower-triangular and $\mathbf{U}$ is upper-triangular, then $\mathbf{LU}$ is diagonal.
False. In general $\mathbf{LU}$ will be a full matrix.

/3 (e) True False $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{C} + \mathbf{A} + \mathbf{B}$.
True, addition is commutative..

/3 (f) True False $\mathbf{ABC} = \mathbf{CAB}$.
False, matrix multiplication is not commutative.

/3 (g) True False If the system $\mathbf{Ax} = \mathbf{b}$ is *consistent*, then it has exactly one solution for $\mathbf{x}$.
False, *consistent* means it has at least one solution.

/3 (h) True False If the system $\mathbf{Ax} = \mathbf{b}$ is *homogeneous*, then it has at least one solution for $\mathbf{x}$.
True, *homogeneous* means $\mathbf{b} = 0$, so $\mathbf{x} = 0$ is a solution.

/3 (i) True False If the set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ is a *basis*, then it is *linearly dependent*.
False, a *basis* (for some other set of vertors) is a *spanning set* that is also *linearly independent*.

/3 (j) True False $\left\| \begin{bmatrix} 1 & 3 & 2 & -3 & 1 & -1 & 0 \end{bmatrix} \right\| = 5$.
True, $\sqrt{1^2 + 3^3 + 2^2 + (-3)^2 + 1^2 + (-1)^2 + 0^2} = \sqrt{1+9+4+9+1+1+0} = \sqrt{25} = 5$.

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & -8 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

/5 (a) Compute $\mathbf{AC} - 2\mathbf{I}$.

$$\begin{bmatrix} 1 & 2 \\ -4 & -8 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -28 & -32 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -28 & -34 \end{bmatrix}$$

/5 (b) Solve $\mathbf{Ax} = \mathbf{b}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -4 & -8 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 7 \end{array}\right].$$

Since $0 = 7$ is false, the system is inconsistent and their are no solutions.

/5 (c) Solve $\mathbf{Ax} = \mathbf{d}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -4 & -8 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

We can let $x_2 = t$ be arbitrary and then $x_1 = 1 - 2t$. In vector form the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Checking,

$$\begin{bmatrix} 1 & 2 \\ -4 & -8 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{d}$$

/5 (d) Solve $\mathbf{Cx} = \mathbf{b}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 5 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 11 & 0 \end{array}\right]$$

so $x_2 = 0$ and $x_1 = 1$. Checking,

$$\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

.

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 7 \\ -5 & -8 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$.

/3 (a) Write the augmented matrix $\mathbf{A}^{\mathbf{b}}$ representing the system $\mathbf{Ax} = \mathbf{b}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 3 & 3 & 7 & 20 \\ -5 & -8 & -7 & 30 \end{array}\right]$$

/12 (b) Use Gaussian Elimination to transform the augmented matrix $\mathbf{A}^{\mathbf{b}}$ to row-reduced form.

Applying the row operations $R_2 \mapsto R_2 - 3R_1$ and $R_3 \mapsto R_3 - (-5)R_1$ yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -2 & -10 \\ 0 & 2 & 8 & 80 \end{array}\right].$$

Applying $R_3 \mapsto R_3 - (-2/3)R_2$ then yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -2 & -10 \\ 0 & 0 & 20/3 & 220/3 \end{array}\right].$$

Applying $R_2 \mapsto (-1/3)R_2$ and $R_3 \mapsto (3/20)R_3$ yields the row-reduced form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2/3 & 10/3 \\ 0 & 0 & 1 & 11 \end{array}\right].$$

/5 (c) Use the row-reduced form of the augmented matrix to solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$.

Working from bottom to top we have

$$\begin{aligned} x_3 &= 11\,, \\ x_2 &= 10/3 - (2/3)11 = (1/3)(10 - 22) = (1/3)(-12) = -4\,, \quad \text{and} \\ x_1 &= 10 - 2(-4) - 3(11) = 10 + 8 - 33 = -15\,. \end{aligned}$$

/5 (d) Multiply $\mathbf{Ax}$ to check that it equals $\mathbf{b}$. Show your work.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 7 \\ -5 & -8 & -7 \end{bmatrix} \begin{bmatrix} -15 \\ -4 \\ 11 \end{bmatrix} &= \begin{bmatrix} 1(-15) + 2(-4) + 3(11) \\ 3(-15) + 3(-4) + 7(11) \\ -5(-15) + (-8)(-4) + (-7)11 \end{bmatrix} \\ &= \begin{bmatrix} -15 - 8 + 33 \\ -45 - 12 + 77 \\ 75 + 32 - 77 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \mathbf{b}\,. \end{aligned}$$

/5 4. Define what it means for a set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ to be *linearly dependent*.

A set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ is *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \cdots + c_n\mathbf{V}_n = 0\,.$$

/20 5. Dang! You are supposed to feed the tissue culture before leaving town for the weekend, but the bottle is empty and everyone else has left already. You normally give it 5ml of a solution that has 2g/ml glucose and 3g/ml sucrose. You find a blue bottle that has 30g/ml glucose and 20g/ml sucrose, and a green bottle that has 10g/ml glucose and 20g/ml sucrose. There is also a supply of distilled water. What do you do?

Let x_1 be the amount of distilled water, x_2 be the amount of solution from the blue bottle, and x_3 be the amount of solution from the green bottle. We want to give it 5ml total liquid, $5\text{ml} \cdot 2\text{g/ml} = 10\text{g}$ glucose, and $5\text{ml} \cdot 3\text{g/ml} = 15\text{g}$ sucrose. Each of these gives an equation that $\mathbf{x}$ needs to satisfy. The system is

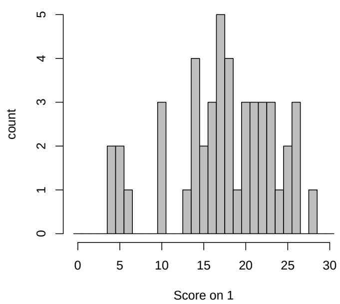
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 30 & 10 & 10 \\ 0 & 20 & 20 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 40/3 & 25/3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 1 & 5/8 \end{array}\right]$$

so $x_3 = 5/8$, $x_2 = 1/3 - (1/3)(5/8) = 1/8$, and $x_1 = 5 - 5/8 - 1/8 = 17/4$. Thus you should mix 17/4ml of distilled water, 1/8ml of the blue bottle, and 5/8ml of the greem bottle, and then feed that to the culture.

As a check,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 30 & 10 \\ 0 & 20 & 20 \end{bmatrix} \begin{bmatrix} 17/4 \\ 1/8 \\ 5/8 \end{bmatrix} = \begin{bmatrix} 17/4 + 1/8 + 5/8 \\ 0 + 30/8 + 50/8 \\ 0 + 20/8 + 100/8 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}.$$

Scores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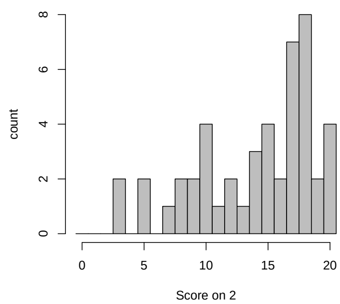
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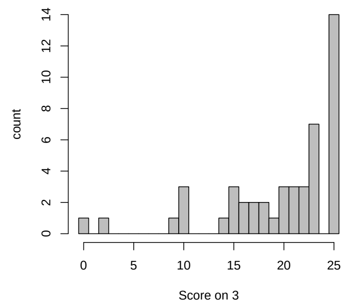
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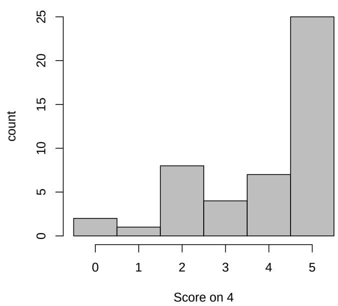
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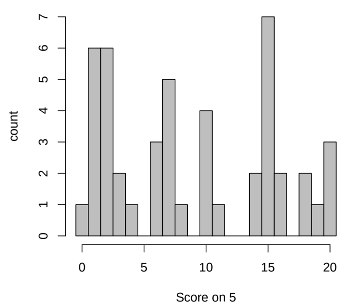
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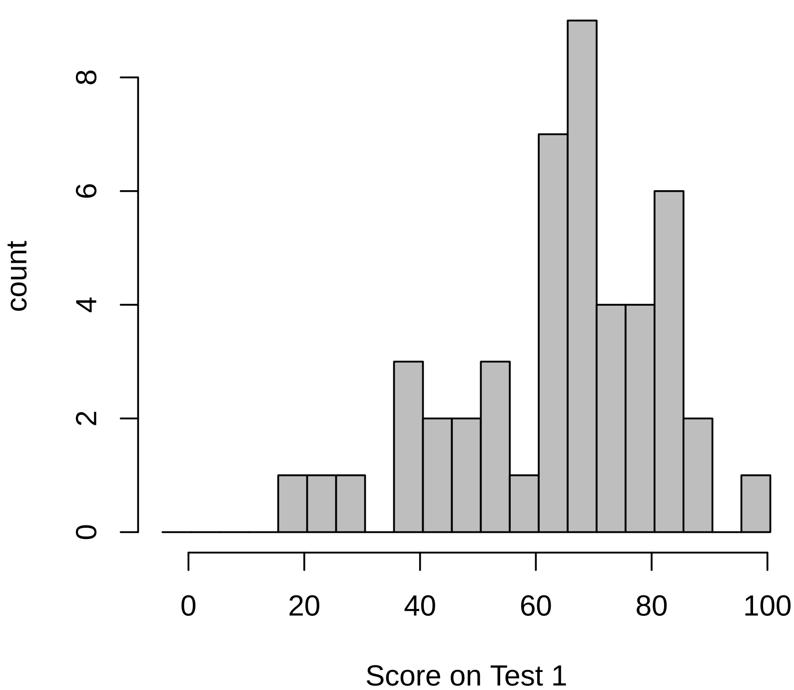
0 5 10 15 20
Score on 5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0 20 40 60 80 100
Score on Test 1