score	possible	page
	30	1
	25	2
	20	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.
Assume that the orders of the matrices are compatible so that they can be added or multiplied.

/3 (a) True False If $\mathbf{A}$ is square, then $\mathbf{A}e^{\mathbf{A}} = e^{\mathbf{A}}\mathbf{A}$.

True. $\mathbf{A}e^{\mathbf{A}} = \mathbf{A}\sum_{k=0}^{\infty}\frac{\mathbf{A}^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathbf{A}^{k+1}}{k!} = \left(\sum_{k=0}^{\infty}\frac{\mathbf{A}^k}{k!}\right)\mathbf{A} = e^{\mathbf{A}}\mathbf{A}$.

/3 (b) True False If $\mathbf{A}$ is square, then $e^{(\mathbf{A}^2)} = \left(e^{\mathbf{A}}\right)^2$.

False. It is false even for numbers. For example $e^{(1^2)} \neq \left(e^1\right)^2$.

/3 (c) True False If $\mathbf{A}(t)$ is square then $\frac{d}{dt}\left(\mathbf{A}(t)\right)^2 = \left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{A}(t) + \mathbf{A}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)$.

True. The product rule works for matrices. (This is property 4 in Section 7.9.)

/3 (d) True False If $\mathbf{A}(t)$ is square then $\frac{d}{dt}\left(\mathbf{A}(t)\right)^2 = 2\mathbf{A}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)$.

False. The power rule does not work for matrices since in general $\mathbf{A}(t)$ and $\frac{d}{dt}\mathbf{A}(t)$ do not compute. (This was exercise 6 in Section 7.9)

/3 (e) True False If $\mathbf{A}$ is square and $\mathbf{B}$ is invertible, then $\left(e^{\mathbf{A}t}\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}e^{-\mathbf{A}t}$.

True. $\left(e^{\mathbf{A}t}\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}\left(e^{\mathbf{A}t}\right)^{-1} = \mathbf{B}^{-1}e^{-\mathbf{A}t}$.

/3 (f) True False If $\mathbf{A}$ is square and t is a scalar, then $\left(e^{\mathbf{A}t}\right)^T = e^{\mathbf{A}^T t}$.

True. This is essentially Section 7.8 Property 3, which you proved in exercise 7. Since t is a scalar, $(\mathbf{A}t)^T = \mathbf{A}^T t$.

/3 (g) True False If $\mathbf{A}$ is diagonalizable, then it is also invertible.

False. A matrix that is all 0 is diagonal already but is not invertible.

/3 (h) True False $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

False. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$, which is not a multiple of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

/3 (i) True False $\mathbf{ABC} = \mathbf{CAB}$.

False, matrix multiplication is not commutative.

/3 (j) True False $\mathbf{AA}^{-1}\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{AA}^{-1}\mathbf{A}$.

True, both equal $\mathbf{I}$.

2. For the vectors $\mathbf{x} = [\ 4 \quad 2 \quad -3\ ]^T$ and $\mathbf{y} = [\ 5 \quad -2 \quad -3\ ]^T$:

/5 (a) Compute $\langle \mathbf{x}, \mathbf{y} \rangle$.

$4(5) + 2(-2) + (-3)(-3) = 20 - 4 + 9 = 25.$

/5 (b) Normalize $\mathbf{x}$.

$\|\mathbf{x}\| = \sqrt{16+4+9} = \sqrt{29}$ so $\frac{\mathbf{x}}{\|\mathbf{x}\|} = [\ 4/\sqrt{29} \quad 2/\sqrt{29} \quad -3/\sqrt{29}\ ]^T$.

/15 3. The matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 9 \\ 0 & -16 & 17 \end{bmatrix}$ has eigenvalues

- $\lambda_1 = 3$ with algebraic multiplicity 1 whose eigenspace has a basis $\left\{ [\ 1 \quad 0 \quad 0\ ]^T \right\}$ and
- $\lambda_2 = 5$ with algebraic multiplicity 2 whose eigenspace has a basis $\left\{ [\ 0 \quad 3 \quad 4\ ]^T \right\}$.

Find a matrix $\mathbf{P}$ that is invertible and a matrix $\mathbf{J}$ that is in Jordan normal form such that $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$. (Remember to **show your work**.)

Since $\lambda_1 = 3$ has algebraic and geometric multiplicity 1, it gives a Jordan block $\mathbf{J}_1 = [\ 3\ ]$. Since $\lambda_2 = 5$ algebraic multiplicity 2 and geometric multiplicity 1, it gives a Jordan block $\mathbf{J}_2 = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. Assembling these gives

$$\mathbf{J} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

To replace the missing linearly independent eigenvector for λ_2, we solve $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = [\ 0 \quad 3 \quad 4\ ]^T$. Using the row-reduction $R_3 \mapsto R_3 - (4/3)R_2$ we obtain

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -12 & 9 & 3 \\ 0 & -16 & 12 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -12 & 9 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which means $x_1 = 0$, x_3 is free, and $x_2 = (3 - 9x_3)/(-12)$. Choosing $x_3 = 0$ gives the vector $[\ 0 \quad -1/4 \quad 0\ ]^T$. Assembling the linearly independent vectors in the order used for $\mathbf{J}$ yields

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1/4 \\ 0 & 4 & 0 \end{bmatrix}.$$

/10 4. The matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Compute $e^{\mathbf{A}t}$.

Matching $e^{xt} = r(xt) = a_1xt + a_0$ at the eigenvalues gives the system

$$\left[\begin{array}{cc|c} 1t & 1 & e^t \\ 4t & 1 & e^{4t} \end{array}\right] \to \left[\begin{array}{cc|c} 1t & 1 & e^t \\ 3t & 0 & e^{4t} - e^t \end{array}\right]$$

so $a_1 = \frac{e^{4t}-e^t}{3t}$ and $a_0 = e^t - t\frac{e^{4t}-e^t}{3t} = \frac{4e^t-e^{4t}}{3}$.

Thus

$$e^{\mathbf{A}t} = \frac{e^{4t}-e^t}{3t}\mathbf{A}t + \frac{4e^t-e^{4t}}{3}\mathbf{I} = \frac{e^{4t}-e^t}{3}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} + \frac{4e^t-e^{4t}}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2\frac{e^{4t}-e^t}{3} + \frac{4e^t-e^{4t}}{3} & 1\frac{e^{4t}-e^t}{3} \\ 2\frac{e^{4t}-e^t}{3} & 3\frac{e^{4t}-e^t}{3} + \frac{4e^t-e^{4t}}{3} \end{bmatrix}.$$

/10 5. The matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ has eigenvalue $\lambda = 3$ with multiplicity 2. Compute $e^{\mathbf{A}}$.

Matching $f(x) = e^x = r(x) = a_1x + a_0$ and $f'(x) = e^x = r'(x) = a_1$ at the eigenvalue gives the system

$$\left[\begin{array}{cc|c} 3 & 1 & e^3 \\ 1 & 0 & e^3 \end{array}\right]$$

so $a_1 = e^3$ and $a_0 = e^3 - 3e^3 = -2e^3$.

Thus

$$e^{\mathbf{A}} = e^3\mathbf{A} - 2e^3\mathbf{I} = \begin{bmatrix} 0 & -1e^3 \\ 1e^3 & 2e^3 \end{bmatrix}.$$

/5 6. State the Cayley-Hamilton Theorem.

There are a couple of equivalent ways to state it:

- A matrix satisfies its own characteristic equation.
- If $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$ then $p(\mathbf{A}) = \mathbf{0}$.

7. You are trying to compute $e^{\mathbf{A}t}$ for a particular matrix $\mathbf{A}$ that has real entries but complex eigenvalues. You get complex-looking coefficients but want them to look real. Use Euler's relations to write the following coefficients without complex numbers:

/5 (a) $a_1 = \dfrac{e^{3t+4ti} - e^{3t-4ti}}{11ti} =$

$$e^{3t}\frac{2}{11t}\frac{e^{4ti} - e^{-4ti}}{2i} = e^{3t}\frac{2}{11t}\sin(4t)$$

/5 (b) $a_0 = \dfrac{e^{3t+5ti} + e^{3t-5ti}}{4} =$

$$\frac{e^{3t}}{2}\frac{e^{5ti} + e^{-5ti}}{2} = \frac{e^{3t}}{2}\cos(5t)$$

/10 8. You did an experiment and collected the data

i	1	2	3	4
x_i	0	1	3	5
y_i	7	8	9	10

. You decide to use a polynomial $p(x)$ to interpolate the data, so it should have $p(x_i) = y_i$ for each i. Set up a linear system to determine the coefficients of $p(x)$ and write the system as an augmented matrix. (Do not solve the system.)

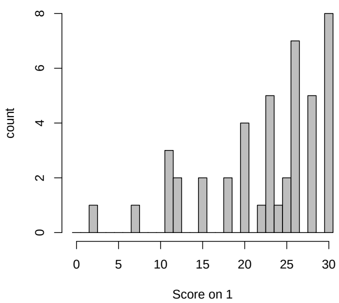
Since there are 4 data points we will have 4 equations, so the polynomial $p(x)$ should have 4 coefficients, which means it is degree 3 and can be written as $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. We get the system of equations

$$\begin{aligned} a_3 0^3 + a_2 0^2 + a_1 0 + a_0 &= 7\,, \\ a_3 1^3 + a_2 1^2 + a_1 1 + a_0 &= 8\,, \\ a_3 3^3 + a_2 3^2 + a_1 3 + a_0 &= 9\,, \quad \text{and} \\ a_3 5^3 + a_2 5^2 + a_1 5 + a_0 &= 10\,. \end{aligned}$$

In augmented matrix form, the system is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 7 \\ 1 & 1 & 1 & 1 & 8 \\ 3^3 & 3^2 & 3 & 1 & 9 \\ 5^3 & 5^2 & 5 & 1 & 10 \end{array}\right].$$

Scores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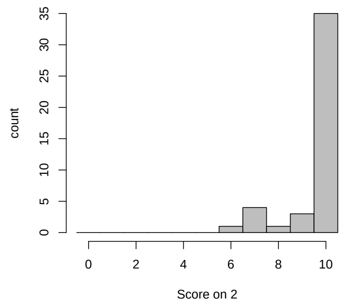
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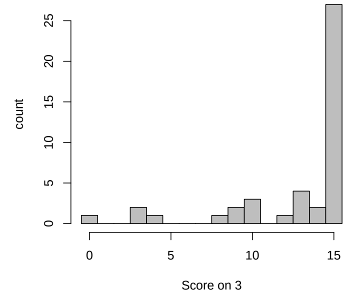
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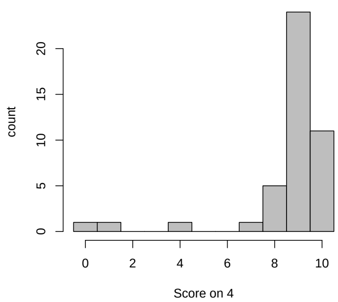
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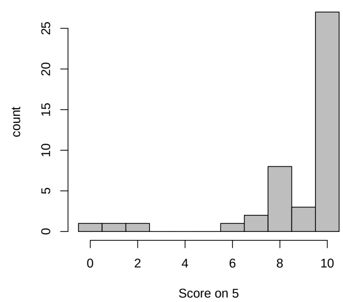
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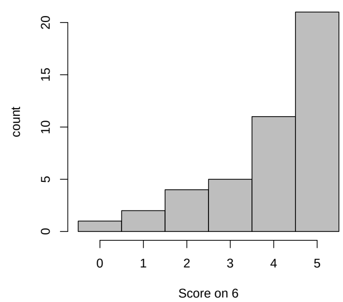
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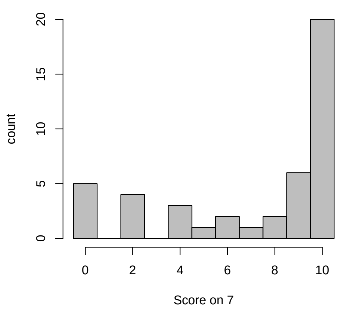
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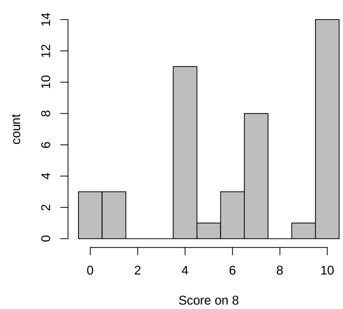
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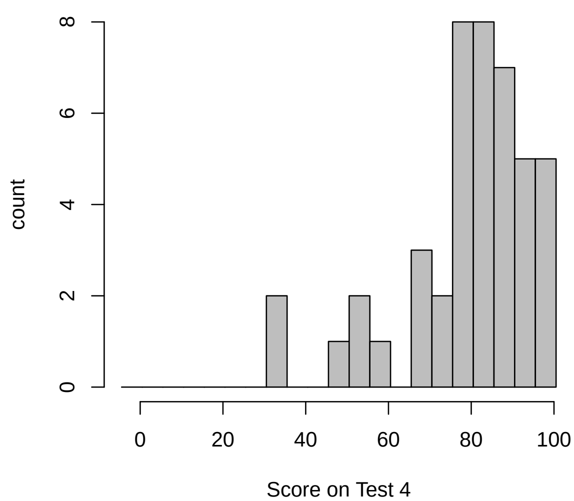
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Score on Test 4