score	possible	page
	30	1
	20	2
	25	3
	30	4
	20	5
	25	6
	150	

Name: ____________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.
Assume that the orders of the matrices are compatible so that they can be added or multiplied.

/3 (a) True False If $\mathbf{L}$ is lower-triangular and $\mathbf{U}$ is upper-triangular, then $\mathbf{LU}$ is diagonal.
False. In general $\mathbf{LU}$ will be a full matrix.

/3 (b) True False If $\mathbf{A}$ is invertible, then $\left((-1/2)\mathbf{A}^{-1}\right)^T$ is invertible.
True, $\left(\left((-1/2)\mathbf{A}^{-1}\right)^T\right)^{-1} = -2\mathbf{A}^T$.

/3 (c) True False If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A}^2$ is similar to $\mathbf{B}^2$.
True. Similar means $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ for some invertible $\mathbf{B}$ so $\mathbf{B}^2 = \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}$, which means $\mathbf{A}^2$ is similar to $\mathbf{B}^2$.

/3 (d) True False If $\mathbf{A}$ is square, then $e^{(\mathbf{A}^2)} = \left(e^{\mathbf{A}}\right)^2$.
False. It is false even for numbers. For example $e^{(1^2)} \neq \left(e^1\right)^2$.

/3 (e) True False If $\mathbf{A}$ is square and you perform the QR-decomposition to get $\mathbf{A} = \mathbf{QR}$, then the diagonal entries of $\mathbf{R}$ are the eigenvalues of $\mathbf{A}$.
False. There is no direct connection.

/3 (f) True False The purpose of the QR-algorithm is to compute the eigenvalues of a matrix $\mathbf{A}$.
True. See Section 10.4.

/3 (g) True False If a set of vectors is *orthonormal*, then it is also *linearly independent*.
True. See Section 10.2 Theorem 1.

/3 (h) True False $\langle \mathbf{x}+\mathbf{z}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}+\mathbf{z}\rangle$.
False. The similar true statement is $\langle \mathbf{x}+\mathbf{z}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{z}, \mathbf{y}\rangle$.

/3 (i) True False For any norm, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
True. This is the triangle inequality, which is one of the defining properties of a norm.

/3 (j) True False The *least-squares* solution to a system $\mathbf{Ax} = \mathbf{b}$ is the solution $\mathbf{x}$ with the smallest value for $\sum_{i=1}^N x_i^2$.
False. It is the $\mathbf{x}$ that minimizes $\|\mathbf{Ax} - \mathbf{b}\|^2$ and is generally not a solution to $\mathbf{Ax} = \mathbf{b}$.

2. You did an experiment and collected the data

i	1	2	3	4
x_i	0	1	3	5
y_i	7	8	9	10

/10 (a) You decide to use a polynomial $p(x)$ to interpolate the data, so it should have $p(x_i) = y_i$ for each i. Set up a linear system to determine the coefficients of $p(x)$ and write the system as an augmented matrix. (Do not solve the system.)

Since there are 4 data points we will have 4 equations, so the polynomial $p(x)$ should have 4 coefficients, which means it is degree 3 and can be written as $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. We get the system of equations

$$\begin{aligned} a_3 0^3 + a_2 0^2 + a_1 0 + a_0 &= 7\,, \\ a_3 1^3 + a_2 1^2 + a_1 1 + a_0 &= 8\,, \\ a_3 3^3 + a_2 3^2 + a_1 3 + a_0 &= 9\,, \quad \text{and} \\ a_3 5^3 + a_2 5^2 + a_1 5 + a_0 &= 10\,. \end{aligned}$$

In augmented matrix form, the system is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 7 \\ 1 & 1 & 1 & 1 & 8 \\ 3^3 & 3^2 & 3 & 1 & 9 \\ 5^3 & 5^2 & 5 & 1 & 10 \end{array}\right].$$

/10 (b) You change your mind and decide to fit the data with a line $p(x)$. Set up a linear system to determine the coefficients of $p(x)$ and write the system as an augmented matrix. (Do not solve the system.)

We can write $p(x) = a_1x + a_0$ and get the (inconsistent) system of equations

$$\begin{aligned} a_1 0 + a_0 &= 7\,, \\ a_1 1 + a_0 &= 8\,, \\ a_1 3 + a_0 &= 9\,, \quad \text{and} \\ a_1 5 + a_0 &= 10\,. \end{aligned}$$

In matrix form $\mathbf{Ax} = \mathbf{b}$ this is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}.$$

Applying $\mathbf{A}^T$ gives the normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, which are

$$\left[\begin{array}{cc|c} 1+9+25 & 1+3+5 & 0+1(8)+3(9)+5(10) \\ 1+3+5 & 4 & 7+8+9+10 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 35 & 9 & 85 \\ 9 & 4 & 34 \end{array}\right].$$

/5 3. Define what it means for a set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ to be *orthonormal.*

For every i, $\|\mathbf{V}_i\| = 1$ and for every $i \neq j$, $\langle \mathbf{V}_i, \mathbf{V}_j \rangle = 0$.

/20 4. Compute the QR-decomposition of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$.

Denote the columns of $\mathbf{A}$ by a_j, the columns of $\mathbf{Q}$ by q_j, and the entries of $\mathbf{R}$ by r_{ij}. Normalizing a_1 gives

$$r_{11} = \|a_1\| = 2 \quad \text{and} \quad q_1 = a_1/r_{11} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Orthogonalizing a_2 to q_1 gives

$$r_{12} = \langle a_2, q_1 \rangle = 2 \quad \text{and} \quad b_2 = a_2 - r_{12}q_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Orthogonalizing a_3 to q_1 gives

$$r_{13} = \langle a_3, q_1 \rangle = 0 \quad \text{and} \quad b_3 = a_3 - r_{13}q_1 = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}.$$

Normalizing b_2 gives

$$r_{22} = \|b_2\| = \sqrt{10} \quad \text{and} \quad q_2 = b_2/r_{22} = \begin{bmatrix} 1/\sqrt{10} \\ 0 \\ 3/\sqrt{10} \end{bmatrix}.$$

Orthogonalizing b_3 to q_2 gives

$$r_{23} = \langle b_3, q_2 \rangle = \sqrt{10} \quad \text{and} \quad c_3 = b_3 - r_{23}q_2 = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}.$$

Normalizing c_3 gives

$$r_{33} = \|c_3\| = 2\sqrt{10} \quad \text{and} \quad q_3 = c_3/r_{33} = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}.$$

Assembling into matrices, we have

$$\mathbf{Q} = \begin{bmatrix} 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & \sqrt{10} & \sqrt{10} \\ 0 & 0 & 2\sqrt{10} \end{bmatrix}.$$

/10 5. (a) Find all the eigenvalues of the matrix $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 3 \\ -1 & 0 & 3 \end{bmatrix}$

Expanding down the second column, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 2 & 3-\lambda & 3 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix}$$
$$= (3-\lambda)((3-\lambda)^2-(-1)^2) = (3-\lambda)(9-6\lambda+\lambda^2-1) = (3-\lambda)(\lambda^2-6\lambda+8) = (3-\lambda)(\lambda-4)(\lambda-2)\,,$$

so we have eigenvalues $\{2, 3, 4\}$.

/10 (b) The matrix $\begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix}$ has eigenvalues 2 and 4. For each eigenvalue, find a basis (of eigenvectors) for the eigenspace.

For each eigenvalue, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. For $\lambda = 2$ this yields

$$\left[\begin{array}{cc|c} 6 & -6 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so x_2 is free and $x_1 = x_2$. Choosing $x_2 = 1$ gives an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 4$ this yields

$$\left[\begin{array}{cc|c} 4 & -6 & 0 \\ 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so x_2 is free and $x_1 = (3/2)x_2$. Choosing $x_2 = 2$ gives an eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

/10 (c) The matrix $\mathbf{A}$ has eigenvalues

- $\lambda_1 = 3$ with algebraic multiplicity 1 whose eigenspace has a basis $\left\{\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T\right\}$ and
- $\lambda_2 = 5$ with algebraic multiplicity 2 whose eigenspace has a basis $\left\{\begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}^T\right\}$.

Find a matrix $\mathbf{P}$ that is invertible and a matrix $\mathbf{D}$ that is diagonal such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Placing the eigenvalues on the diagonal yields

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Placing the corresponding eigenvectors as columns in $\mathbf{P}$ yields

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}.$$

6. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 7 \\ -5 & -8 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$.

/15 (a) Write the augmented matrix $\mathbf{A}^{\mathbf{b}}$ representing the system $\mathbf{Ax} = \mathbf{b}$ and then use Gaussian Elimination to transform the augmented matrix $\mathbf{A}^{\mathbf{b}}$ to row-reduced form.

$$\mathbf{A}^{\mathbf{b}} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 3 & 3 & 7 & 20 \\ -5 & -8 & -7 & 30 \end{array}\right]$$

Applying the row operations $R_2 \mapsto R_2 - 3R_1$ and $R_3 \mapsto R_3 - (-5)R_1$ yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -2 & -10 \\ 0 & 2 & 8 & 80 \end{array}\right].$$

Applying $R_3 \mapsto R_3 - (-2/3)R_2$ then yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -2 & -10 \\ 0 & 0 & 20/3 & 220/3 \end{array}\right].$$

Applying $R_2 \mapsto (-1/3)R_2$ and $R_3 \mapsto (3/20)R_3$ yields the row-reduced form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2/3 & 10/3 \\ 0 & 0 & 1 & 11 \end{array}\right].$$

/5 (b) Use the row-reduced form of the augmented matrix to solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$.

Working from bottom to top we have

$$\begin{aligned} x_3 &= 11\,, \\ x_2 &= 10/3 - (2/3)11 = (1/3)(10-22) = (1/3)(-12) = -4\,, \quad \text{and} \\ x_1 &= 10 - 2(-4) - 3(11) = 10 + 8 - 33 = -15\,. \end{aligned}$$

In vector form we have

$$\mathbf{x} = \begin{bmatrix} -15 \\ -4 \\ 11 \end{bmatrix}.$$

/5 7. For $\mathbf{x} = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, compute

(a) $\|\mathbf{x}\|_1 = 1 + 4 + 3 = 8$

(b) $\|\mathbf{x}\|_2 = \sqrt{1 + 16 + 9} = \sqrt{26}$

(c) $\|\mathbf{x}\|_\infty = 4$

/10 8. Find all solutions to the system of equations $x_1 + 2x_2 + 3x_3 = 4$ and $x_2 + 5x_3 = 6$.

In augmented matrix form, the system is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \end{array}\right]$. Since $\text{rank}(\mathbf{A}) = 2 = \text{rank}(\mathbf{A^b})$, the system is consistent. Since $\mathbf{A}$ has 3 columns there are 3 variables and so $3 - \text{rank}(\mathbf{A}) = 1$ arbitrary parameter. Letting $t = x_3$ be this parameter, we have $x_2 = 6 - 5t$ and $x_1 = 4 - 2(6 - 5t) - 3t = -8 + 7t$. In vector form this is $\mathbf{x} = \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$. We can check

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -8 + 12 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 - 10 + 3 \\ -5 + 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}.$$

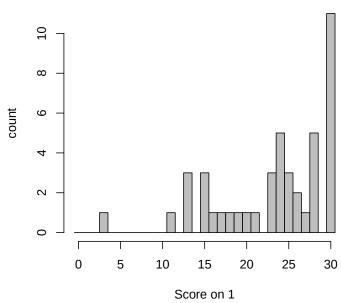
/10 9. A mining company has a contract to supply 70,000 tons of low-grade ore, 181,000 tons of medium-grade ore, and 41,000 tons of high-grade ore to a supplier. The company has three mines which it can work. Mine A produces 8000 tons of low-grade ore, 5000 tons of medium-grade ore, and 1000 tons of high-grade ore during each day of operation. Mine B produces 3000 tons of low-grade ore, 12,000 tons of medium-grade ore, and 3000 tons of high-grade ore for each day it is in operation. The figures for mine C are 1000, 10,000, and 2000, respectively.

Find a system of equations that determines how many days each mine should be operated to meet contractual demands without surplus. Identify the meaning of each variable and the meaning of each equation. (Do not solve the system.)

The variables are a, the number of days mine A operates, b, the number of days mine B operates, and c, the number of days mine C operates. To meet the contract for low-grade ore we must have $8000a + 3000b + 1000c = 70000$, to meet the contract for medium-grade ore we must have $5000a + 12000b + 10000c = 181000$, and to meet the contract for high-grade ore we must have $1000a + 3000b + 2000c = 41000$. In matrix form we have

$$\begin{bmatrix} 8000 & 3000 & 1000 \\ 5000 & 12000 & 10000 \\ 1000 & 3000 & 2000 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 70000 \\ 181000 \\ 41000 \end{bmatrix}.$$

Scores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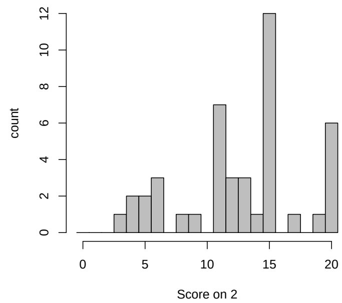
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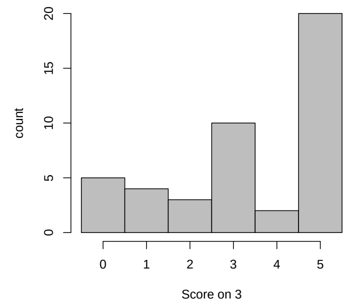
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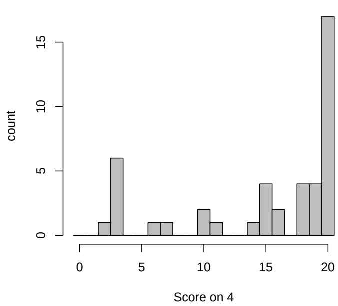
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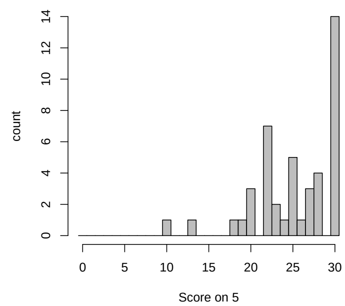
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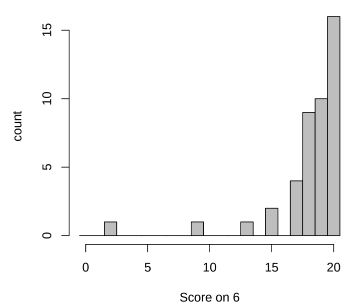
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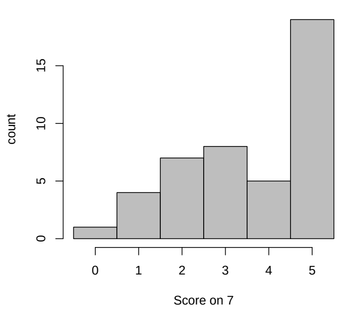
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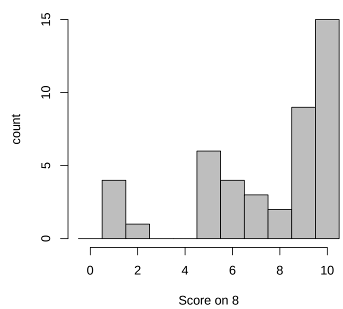
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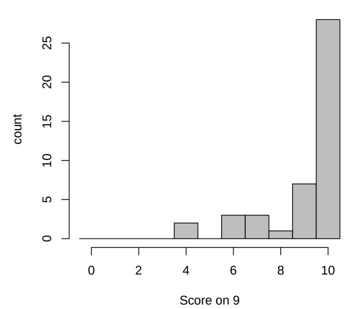
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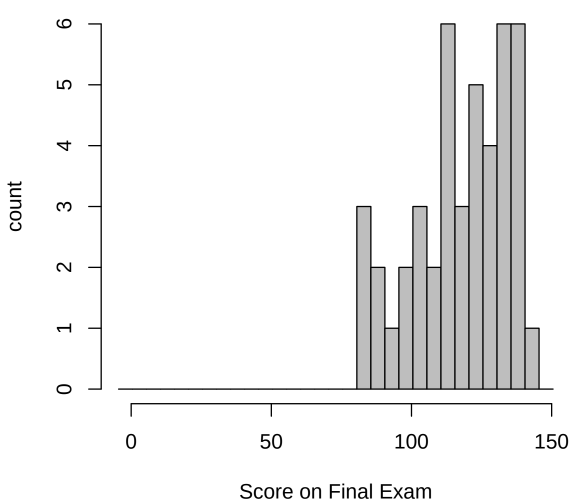
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