1. A farmer wants a rectangular pen next to a large barn, using the wall of the barn as one side of the pen. If the farmer wants the area enclosed to be 1,800 m², what are the dimensions of the fence that minimize the length of fencing used? Make a sketch of the fence and barn, clearly showing the variables you are using.

We want to minimize $P = 2x + y$ and we have the constraint $xy = 1800$ m².

Solving the constraint for $y$ yields $y = x^{-1}1800$ m². Substituting into $P$ gives $P(x) = 2x + x^{-1}1800$ m².

Differentiating gives $P'(x) = 2 - x^{-2}1800$ m² and setting $P'(x) = 0$ gives

$$2 - x^{-2}1800 = 0 \implies 2x^2 = 1800 \implies x = \pm 30 \text{ m}$$

as the critical numbers. Since $x$ must be positive, we only use $x = 30$ m. Differentiating $P'(x)$ gives $P''(x) = 2x^{-3}1800$ m², which is positive for $x > 0$, so this critical number is a local minimum.

Substituting back into the constraint gives $y = (30 \text{ m})^{-1}1800$ m² = 60 m. Thus the side parallel to the barn should be 60 m and the sides perpendicular to the barn should be 30 m.
2. Use Newton’s method with the initial approximation $x_1 = 1$ to find $x_2$, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Leave the answer as a fraction.

Setting $f(x) = x^3 + x + 3$ gives $f'(x) = 3x^2 + 1$. Newton’s method gives the next approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1^3 + 1 + 3}{3(1^2) + 1} = 1 - \frac{5}{4} = -\frac{1}{4}.$$

3. If $f'''(t) = \cos(t)$, find $f(x)$.

Antidifferentiating three times gives

$$f'''(t) = \sin(t) + C,$$
$$f''(t) = -\cos(t) + Ct + D,$$ and $$f'(t) = -\sin(t) + Ct^2/2 + Dt + E,$$

where $C$, $D$, and $E$ are unknown constants.

4. The velocity of a runner increased steadily during the first three seconds of a race. Her velocity at half second intervals is given in the table. Find good upper and lower estimates for the distance that she traveled during these three seconds. Do not simplify.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (ft/s)</td>
<td>0</td>
<td>6.2</td>
<td>10.8</td>
<td>14.9</td>
<td>18.1</td>
<td>19.4</td>
<td>20.2</td>
</tr>
</tbody>
</table>

Since her velocity is increasing, using the left edge of each time interval will give a lower estimate and using the right edge will give an upper estimate. The lower estimate is

$$\frac{1}{2} (0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4) \text{ ft}$$

and the upper estimate is

$$\frac{1}{2} (6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) \text{ ft}.$$
5. Using the definition of the derivative as a limit, compute the derivative of \( f(x) = x^{-1} \).
(Do not use L'Hôpital's rule or differentiation formulas like the power rule or chain rule).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^{-1} - x^{-1}}{h}
= \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{x + h} - \frac{1}{x} \right)
= \lim_{h \to 0} \frac{1}{h} \frac{x - (x + h)}{x(x + h)}
= \lim_{h \to 0} \frac{-1}{x(x + h)}
= \frac{-1}{x(x + 0)} = -\frac{1}{x^2}.
\]

6. Each side of a cube is increasing at a rate of 6 cm/s. At what rate is the volume of the cube changing when the volume is 8 cm\(^3\)?

Let \( x \) denote the length of a side of the cube. The volume is \( V = x^3 \). We are told \( \frac{dx}{dt} = 6 \text{ cm/s} \) and want to find \( \frac{dV}{dt} \) when \( V = 8 \text{ cm}^3 \), which means when \( x = 2 \text{ cm} \).

Differentiating \( V = x^3 \) with respect to \( t \) and plugging in gives

\[
\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = 3(2 \text{ cm})^2(6 \text{ cm/s}) = 72 \text{ cm}^3/\text{s}.
\]
7. For the function \( f(x) = \frac{x}{x^2 - 4} \)

/2 (a) Find the x- and y-intercepts.

/6 (b) Find any asymptotes.

/6 (c) Find the intervals on which \( f \) is increasing or decreasing.

/2 (d) Find the local maximum and minimum values of \( f \).

/6 (e) Find the intervals of concavity and the inflection points.

/8 (f) Use the information above to sketch the graph.

\( f(0) = 0 \) and no other \( x \) makes \( f(x) = 0 \), so both intercepts are at \((0, 0)\).

The denominator is 0 and there are vertical asymptotes at \( x = -2 \) and \( x = 2 \).

\[
\lim_{x \to \pm\infty} \frac{x}{x^2 - 4} = \lim_{x \to \pm\infty} \frac{x}{x^2} = \lim_{x \to \pm\infty} \frac{1}{x} = 0
\]

so there is a horizontal asymptote at \( y = 0 \).

\[
f'(x) = \frac{1(x^2-4)-x(2x)}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}, \text{ which is undefined at } x = \pm 2 \text{ but is never } 0.
\]

\[
f''(x) = \frac{-2x(x^2-4)^2-(-x^2-4)2(x^2-4)2x}{(x^2-4)^4} = \frac{2x(-x^2+4+2x^2+8)}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x^2-4)^3}, \text{ which is undefined at } x = \pm 2 \text{ and } 0 \text{ at } x = 0.
\]

Assembling into a chart and checking signs, we have

<table>
<thead>
<tr>
<th>( f'' )</th>
<th>V.A.</th>
<th>I.P.</th>
<th>V.A.</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>DNE</td>
<td>+</td>
<td>DNE</td>
</tr>
<tr>
<td>( f' )</td>
<td>-</td>
<td>DNE</td>
<td>+</td>
</tr>
<tr>
<td>( x )</td>
<td>(-\infty, -2)</td>
<td>-2</td>
<td>(-2, 0)</td>
</tr>
</tbody>
</table>

The are no local maxima or minima. There is an inflection point at \((0, 0)\).
Scores