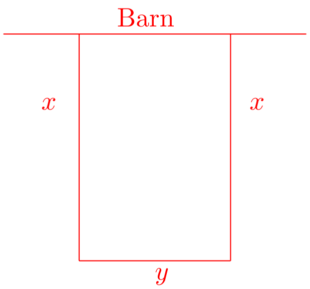
score	possible	page
	20	1
	30	2
	20	3
	30	4
	100	

Name: ________________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/20 1. A farmer wants a rectangular pen next to a large barn, using the wall of the barn as one side of the pen. If the farmer wants the area enclosed to be $1,800\,\mathrm{m}^2$, what are the dimensions of the fence that minimize the length of fencing used? Make a sketch of the fence and barn, clearly showing the variables you are using.



We want to minimize $P = 2x + y$ and we have the constraint $xy = 1800\,\mathrm{m}^2$.

Solving the constraint for y yields $y = x^{-1}1800\,\mathrm{m}^2$. Substituting into P gives $P(x) = 2x + x^{-1}1800\,\mathrm{m}^2$.

Differentiating gives $P'(x) = 2 - x^{-2}1800\,\mathrm{m}^2$ and setting $P'(x) = 0$ gives

$$2 - x^{-2}1800\,\mathrm{m}^2 = 0 \quad \Rightarrow \quad 2x^2 = 1800\,\mathrm{m}^2 \quad \Rightarrow \quad x = \pm 30\,\mathrm{m}$$

as the critical numbers. Since x must be positive, we only use $x = 30\,\mathrm{m}$. Differentiating $P'(x)$ gives $P''(x) = 2x^{-3}1800\,\mathrm{m}^2$, which is positive for $x > 0$, so this critical number is a local minimum.

Substituting back into the constraint gives $y = (30\,\mathrm{m})^{-1}1800\,\mathrm{m}^2 = 60\,\mathrm{m}$. Thus the side parallel to the barn should be $60\,\mathrm{m}$ and the sides perpendicular to the barn should be $30\,\mathrm{m}$.

/10 2. Use Newton's method with the initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Leave the answer as a fraction.

Setting $f(x) = x^3 + x + 3$ gives $f'(x) = 3x^2 + 1$. Newton's method gives the next approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1^3 + 1 + 3}{3(1^2) + 1} = 1 - \frac{5}{4} = -\frac{1}{4}\,.$$

/10 3. If $f'''(t) = \cos(t)$, find $f(x)$.

Antidifferentiating three times gives

$$\begin{aligned} f''(t) &= \sin(t) + C\,, \\ f'(t) &= -\cos(t) + Ct + D\,, \quad \text{and} \\ f(t) &= -\sin(t) + C\frac{t^2}{2} + Dt + E\,, \end{aligned}$$

where C, D, and E are unknown constants.

/10 4. The velocity of a runner increased steadily during the first three seconds of a race. Her velocity at half second intervals is given in the table. Find good upper and lower estimates for the distance that she traveled during these three seconds. Do not simplify.

t (s)	0	.5	1	1.5	2	2.5	3
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

Since her velocity is increasing, using the left edge of each time interval will give a lower estimate and using the right edge will give an upper estimate. The lower estimate is

$$\frac{1}{2}\,(0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4) \text{ ft}$$

and the upper estimate is

$$\frac{1}{2}\,(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) \text{ ft}\,.$$

/10 5. Using the **definition of the derivative as a limit**, compute the derivative of $f(x) = x^{-1}$. (Do not use L'Hôpital's rule or differentiation formulas like the power rule or chain rule).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^{-1} - x^{-1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= \frac{-1}{x(x+0)} = \frac{-1}{x^2}\,. \end{aligned}$$

/10 6. Each side of a cube is increasing at a rate of $6\,\text{cm/s}$. At what rate is the volume of the cube changing when the volume is $8\,\text{cm}^3$?

Let x denote the length of a side of the cube. The volume is $V = x^3$. We are told $\frac{dx}{dt} = 6\,\text{cm/s}$ and want to find $\frac{dV}{dt}$ when $V = 8\,\text{cm}^3$, which means when $x = 2\,\text{cm}$.

Differentiating $V = x^3$ with respect to t and plugging in gives

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = 3(2\,\text{cm})^2(6\,\text{cm/s}) = 72\text{cm}^3/\text{s}\,.$$

7. For the function
$$f(x) = \frac{x}{x^2-4}$$

/2 (a) Find the x- and y-intercepts.

/6 (b) Find any asymptotes.

/6 (c) Find the intervals on which f is increasing or decreasing.

/2 (d) Find the local maximum and minimum values of f.

/6 (e) Find the intervals of concavity and the inflection points.

/8 (f) Use the information above to sketch the graph.

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is 0 and there are vertical asymptotes at $x = -2$ and $x = 2$.

$\lim_{x\to\pm\infty} \frac{x}{x^2-4} = \lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

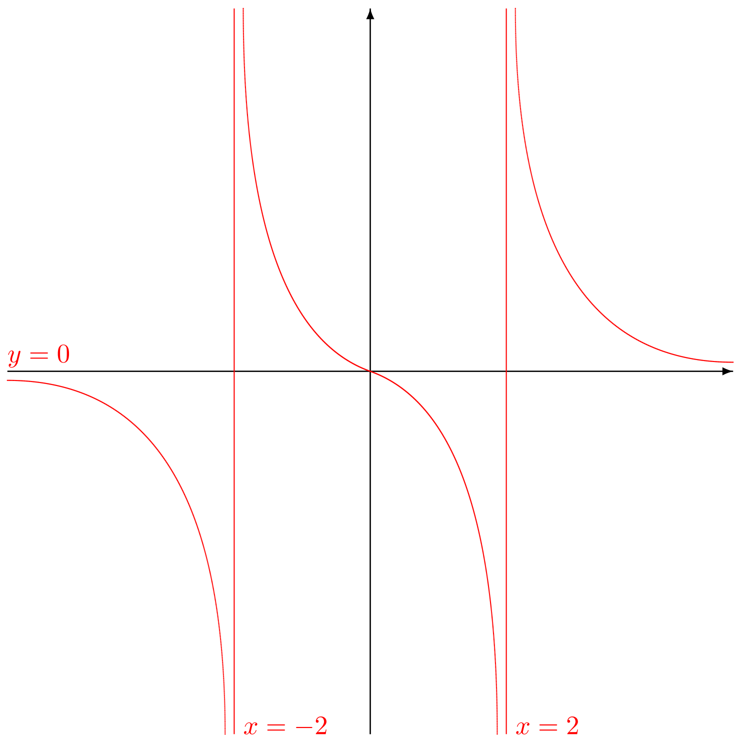
$f'(x) = \frac{1(x^2-4)-x(2x)}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}$, which is undefined at $x = \pm 2$ but is never 0.

$f''(x) = \frac{-2x(x^2-4)^2-(-x^2-4)2(x^2-4)2x}{(x^2-4)^4} = \frac{-2x(x^2-4)-(-x^2-4)4x}{(x^2-4)^3} = \frac{2x(-x^2+4+2x^2+8)}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x^2-4)^3}$, which is undefined at $x = \pm 2$ and 0 at $x = 0$.

Assembling into a chart and checking signs, we have

f	⌒ (concave down)	V.A	◡ (concave up)	I.P.	⌒ (concave down)	V.A.	◡ (concave up)
f''	$-$	DNE	$+$	0	$-$	DNE	$+$
f'	$-$	DNE	$-$	$-$	$-$	DNE	$-$
	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,\infty)$

The are no local maxima or minima. There is an inflection point at $(0, 0)$.



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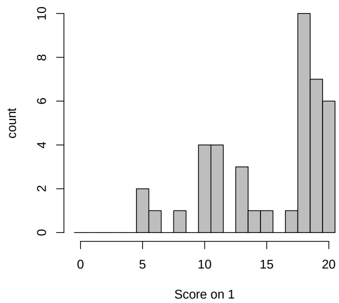
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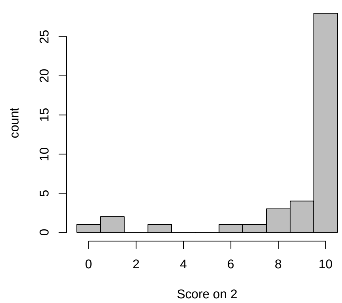
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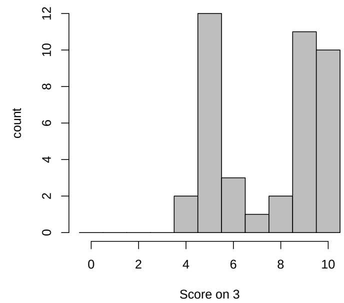
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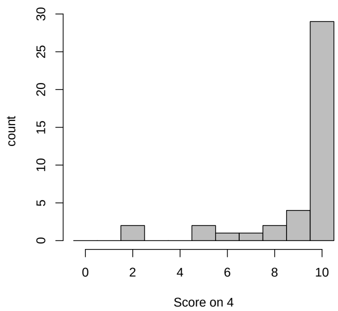
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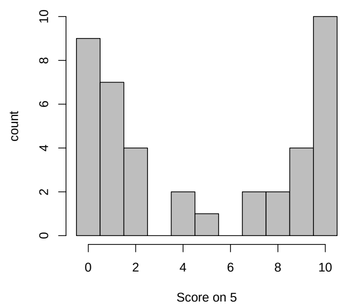
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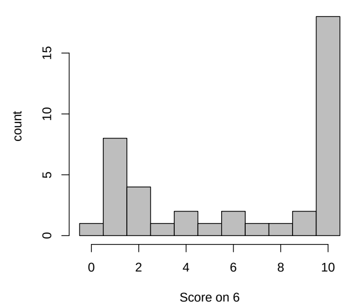
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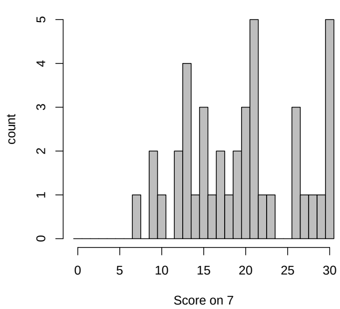
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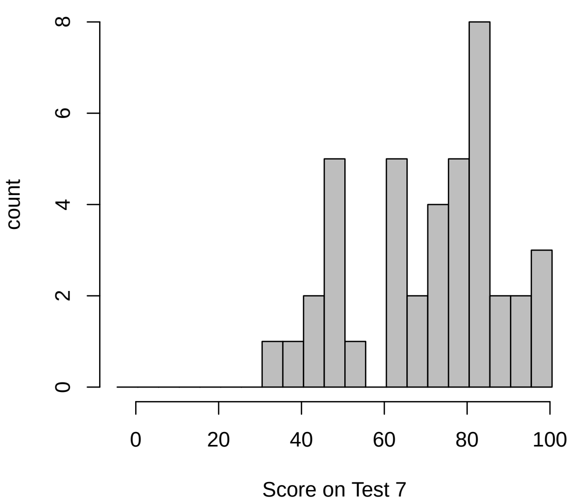
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Score on Test 7