1. Sketch the graph of a single function $f$ that has all of the following properties:

(a) $f$ has a local maximum at $x = 0$ but is not differentiable there.

(b) $\lim_{x \to 2^+} f(x) = -\infty$.

(c) $\lim_{x \to 2^-} f(x) = \infty$.

(d) $f$ is continuous except at $x = 2$.

(e) $f$ has no inflection points.

(f) $\lim_{x \to +\infty} f(x) = \infty$.

(g) $\lim_{x \to -\infty} f(x) = -2$.

We are only given partial information about the function, and need to deduce more.

Since $\lim_{x \to 2^+} f(x) = -\infty$, on the interval $(2, ?)$ $f$ must be increasing and concave down. Since it has no inflection points and $\lim_{x \to +\infty} f(x) = \infty$, it must stay increasing and concave down on $(2, \infty)$.

Since $\lim_{x \to 2^-} f(x) = \infty$, on the interval $(?, 2)$ $f$ must be increasing and concave up. Since it has no inflection points and a local max at $x = 0$, on the interval $(0, ?)$ $f$ must be decreasing and concave up and on $(?, 0)$ it must be increasing and concave up. In order to have $\lim_{x \to -\infty} f(x) = -2$ without an inflection point, on $(-\infty, 0)$ it must be increasing and concave up.

Organizing into a chart, we have

\[
\begin{array}{cccccc}
& f'' & \text{cusp} & \rightarrow & \text{V.A.} & \\
f' & + & \text{DNE} & + & + & \text{DNE} & - \\
\text{at} & (-\infty, 0) & 0 & (0, ?) & ? & (?, 2) & 2 & (2, \infty)
\end{array}
\]

(Other features, such as more points where $f$ is not differentiable, are possible but not needed.)
2. Let \( f(x) = 2x^3 + 3x^2 - 36x \).

(a) Find the intervals where \( f \) is increasing, and the intervals where it is decreasing.

\[
f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2) \text{ so the critical numbers are } x = -3 \text{ and } x = 2.
\]

The sign chart is:

<table>
<thead>
<tr>
<th></th>
<th>(-\infty, -3)</th>
<th>-3</th>
<th>(-3, 2)</th>
<th>2</th>
<th>(2, \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f')</td>
<td>↑</td>
<td>→</td>
<td>↘</td>
<td>→</td>
<td>↑</td>
</tr>
</tbody>
</table>

so \( f \) is increasing on \((-\infty, -3)\) and \((2, \infty)\) and decreasing on \((-3, 2)\).

(b) Find the intervals where \( f \) is concave up, and the intervals where it is concave down.

\[
f''(x) = 6(2x + 1) \text{ so } f''(x) = 0 \text{ at } x = -1/2.
\]

The sign chart is:

<table>
<thead>
<tr>
<th></th>
<th>(-\infty, -1/2)</th>
<th>(-1/2)</th>
<th>(-1/2, \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f'')</td>
<td>−</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

so \( f \) is concave up on \((1/2, \infty)\) and concave down on \((-\infty, -1)\).

(c) Find the absolute maximum and minimum values of \( f \) on the interval \([0, 3]\).

The only critical number in the interval is \( x = 2 \). Evaluating there and at the endpoints we get:

\[
\begin{align*}
    f(0) &= 0, \\
    f(2) &= 2(8) + 3(4) - 36(2) = 16 + 12 - 72 = -44, \text{ and} \\
    f(3) &= 2(27) + 3(9) - 36(3) = 54 + 27 - 108 = -27.
\end{align*}
\]

Thus the absolute maximum is 0 and occurs at \( x = 0 \) and the absolute minimum is -44 and occurs at \( x = 3 \).
3. For the function \( f(x) = \frac{x}{x^2 + 4} \)

/2  (a) Find the \( x \)- and \( y \)-intercepts.

/4  (b) Find any asymptotes.

/6  (c) Find the intervals on which \( f \) is increasing or decreasing.

/4  (d) Find the local maximum and minimum values of \( f \).

/8  (e) Find the intervals of concavity and the inflection points.

/6  (f) Use the information above to sketch the graph.

(\( f \) has odd symmetry, so we could save half the work, but this is optional.)

\( f(0) = 0 \) and no other \( x \) makes \( f(x) = 0 \), so both intercepts are at \((0, 0)\).

The denominator is never 0 so there are no vertical asymptotes.

\[
\lim_{x \to \pm \infty} \frac{x}{x^2 + 4} = \lim_{x \to \pm \infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{4}{x^2}} = \lim_{x \to \pm \infty} \frac{1}{x} = 0 \quad \text{so there is a horizontal asymptote at} \quad y = 0.
\]

\[
f'(x) = \frac{1(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{-x^2 + 4}{(x^2 + 4)^2} = \frac{(2 + x)(2 - x)}{(x^2 + 4)^2} \quad \text{which is zero at} \quad x = -2 \quad \text{and} \quad x = 2.
\]

\[
f''(x) = \frac{-2x(x^2 + 4)^2 - (x^2 + 4)^2(2x^2 + 4)2x}{(x^2 + 4)^4} = \frac{-2x(x^2 + 4)(-x^2 + 4)}{(x^2 + 4)^3} = \frac{2x(x^2 - 4 + 2x^2 - 8)}{(x^2 + 4)^3} = \frac{2x(x^2 - 12)}{(x^2 + 4)^3} = \frac{2x(x + \sqrt{12})(x - \sqrt{12})}{(x^2 + 4)^3},
\]

which is 0 at \( x = 0, x = -2\sqrt{3}, \) and \( x = 2\sqrt{3} \).

Assembling into a chart and checking signs, we have

\[
\begin{array}{cccccccc}
\hline
x & f' & f'' & I.P. & (\text{I.P.}) & I.P. & I.P. \\
\hline
\infty & - & 0 & + & + & + & 0 & + \\
-2\sqrt{3} & - & & & & & & \\
-2 & - & & & & & & \\
-2\sqrt{3}/2 & - & & & & & & \\
0 & - & & & & & & \\
2 & - & & & & & & \\
2\sqrt{3} & - & & & & & & \\
\infty & - & & & & & & \\
\end{array}
\]

The is a local max at \( x = 2 \) with value \( f(2) = 2/(2^2 + 4) = 1/4 \) and a local min at \( x = -2 \) with value \( f(-2) = -2/(2^2 + 4) = -1/4 \). There are inflection points at \((-2\sqrt{3}, f(-2\sqrt{3}) = (-2\sqrt{3}, -2\sqrt{3}/(12 + 4)) = (-2\sqrt{3}, -\sqrt{3}/8)), (0, 0), \) and \((2\sqrt{3}, f(2\sqrt{3}) = (2\sqrt{3}, \sqrt{3}/8)).\)
4. State the Mean Value Theorem (MVT).

If

- $f$ is continuous on the closed interval $[a, b]$ and
- $f$ is differentiable on the open interval $(a, b)$,

then there exists $c \in (a, b)$ such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]

5. Compute $\lim_{x \to \infty} 2x e^{3x} = \infty$. Plugging in to $2x e^{3x}$ gives a $\infty/\infty$ indeterminate form, so we can apply L’Hôpital’s rule to get $\lim_{x \to \infty} \frac{6e^{3x}}{1} = \infty$.

6. Use a linear approximation (or differentials) to estimate $(1.99)^4$.

Set $f(x) = x^4$ so $f'(x) = 4x^3$. Selecting $a = 2$ we have the linear approximation

\[ f(x) \approx L_2(x) = f(2) + f'(2)(x - 2) = 16 + 32(x - 2) \]

so $(1.99)^4 = f(1.99) \approx 16 + 32(-0.01) = 16 - 0.32 = 15.68$.

7. For the function $f(x) = 1 + \sqrt{x}$, find the equation for the tangent line at $x = 4$.

We can compute $f'(x) = \frac{1}{2\sqrt{x}}$. The general form of the tangent line at $A$ is $y = f'(A)(x - A) + f(A)$ so we have

\[ y = f'(4)(x - 4) + f(4) = \frac{1}{2\sqrt{4}}(x - 4) + (1 + \sqrt{4}) = \frac{1}{4}(x - 4) + 3. \]
Scores

Score on 1

Score on 2a

Score on 2b

Score on 2c

Score on 3

Score on 4

Score on 5

Score on 6

Score on 7

Score on Test 6