score	possible	page
	30	1
	30	2
	20	3
	20	4
	100	

Name: ________________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Compute the following derivatives.

/3 (a) $\frac{d}{dx}\left[x^5\right] = 5x^4$

/3 (b) $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

/3 (c) $\frac{d}{dx}[5^x] = 5^x\ln(5)$

/3 (d) $\frac{d}{dx}[\operatorname{arcsec}(x)] = \frac{1}{x\sqrt{x^2-1}}$

/3 (e) $\frac{d}{dx}\left[5^5\right] = 0$

/3 (f) $\frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$

/3 (g) $\frac{d}{dx}\left[x^{1/5}\right] = (1/5)x^{-4/5}$

/3 (h) $\frac{d}{dx}\left[x^{-5}\right] = -5x^{-6}$

/3 (i) $\frac{d}{dx}[\log_5(x)] = \frac{1}{x\ln(5)}$

/3 (j) $\frac{d}{dx}[x\ln(x)] = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1$

2. Compute the following limits. If you use the Squeeze theorem or L'Hôpital's rule, then say so.

/3 (a) $\lim_{x\to 0^+} e^x = e^0 = 1$

/3 (b) $\lim_{x\to\infty} e^x = \infty$

/3 (c) $\lim_{x\to-\infty} \arctan(x) = -\pi/2$

/3 (d) $\lim_{x\to\infty} \tanh(x) = 1$

/3 (e) $\lim_{x\to 0^+} \frac{2}{x}\sin(3x) =$

$\lim_{x\to 0^+} 6\frac{\sin(3x)}{3x} = \lim_{t\to 0^+} 6\frac{\sin(t)}{t} = 6$ if we remember the last limit. Alternatively, if we note that plugging in to $\frac{2\sin(3x)}{x}$ gives a $0/0$ indeterminate form, we can apply L'Hôpital's rule to get $\lim_{x\to 0^+} \frac{6\cos(3x)}{1} = 6$.

/3 (f) $\lim_{x\to 0^+} \frac{2}{x}e^{3x} = 2e^0 \lim_{x\to 0^+} \frac{1}{x} = 2\infty = \infty$

/3 (g) $\lim_{x\to\infty} \frac{2}{x}e^{3x} =$

Plugging in to $\frac{2e^{3x}}{x}$ gives a ∞/∞ indeterminate form, so we can apply L'Hôpital's rule to get $\lim_{x\to\infty} \frac{6e^{3x}}{1} = \infty$.

/3 (h) $\lim_{x\to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$

/3 (i) $\lim_{x\to 0^+} x\ln(x) =$

Plugging in yields $0^+(-\infty)$, which is indeterminate but not in the form for L'Hôpital's rule. Rewriting to get $-\infty/\infty$ form, we can apply L'Hôpital's rule to get

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0\,.$$

/3 (j) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} =$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}\frac{x^{-1}}{x^{-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+x^{-2}}} = \frac{1}{\sqrt{1+0}} = 1$$

/10 3. Use logarithmic differentiation to compute the derivative of $y = \dfrac{x^x \sin(2x)}{3^x\sqrt{x^9+1}}$.

Applying the natural logarithm to both sides and using rules of logarithms yields

$$\ln(y) = x\ln(x) + \ln(\sin(2x)) - x\ln(3) - \frac{1}{2}\ln(x^9+1)\,.$$

Differentiating both sides yields

$$\frac{y'}{y} = \ln(x) + 1 + \frac{\cos(2x)2}{\sin(2x)} - \ln(3) - \frac{1}{2}\frac{9x^8}{x^9+1}\,.$$

Solving for y' then gives

$$y' = \left(\ln(x) + 1 + \frac{\cos(2x)2}{\sin(2x)} - \ln(3) - \frac{1}{2}\frac{9x^8}{x^9+1}\right)\frac{x^x\sin(2x)}{3^x\sqrt{x^9+1}}\,.$$

/10 4. Given that $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$, show that $\dfrac{d}{dx}\arctan(\tanh(x)) = \operatorname{sech}(2x)$.

Computing the derivative on the left yields

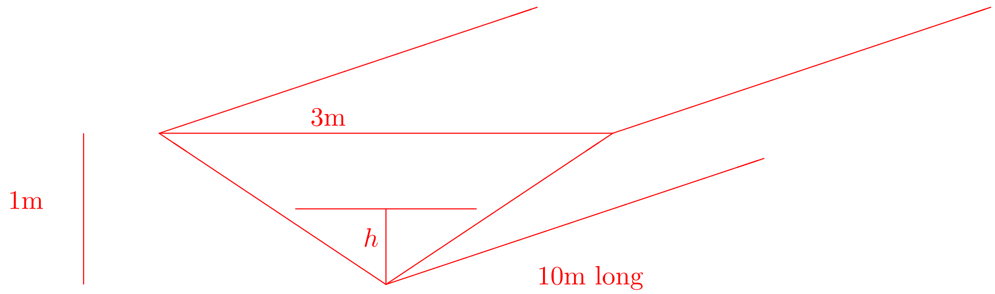
$$\frac{d}{dx}\arctan(\tanh(x)) = \frac{1}{1+\tanh^2(x)}\operatorname{sech}^2(x) = \frac{1}{1+\left(\frac{\sinh(x)}{\cosh(x)}\right)^2}\frac{1}{\cosh^2(x)} = \frac{1}{\cosh^2(x)+\sinh^2(x)}\,.$$

Using the given identity, the right side yields

$$\operatorname{sech}(2x) = \frac{1}{\cosh(2x)} = \frac{1}{\cosh(x+x)} = \frac{1}{\cosh^2(x)+\sinh^2(x)}\,.$$

Since these agree, we have shown the identity.

/10 5. A trough is 10m long and its ends have the shape of isosceles triangles that are 3m across at the top and have a height of 1m. The trough is being filled with water at a rate of $12\text{m}^3/\text{min}$. Draw and label a diagram illustrating this scenario. How fast is the water level rising when it is 0.5m deep?



Since the triangular end has proportion 3/1, for a given height of water h the width is $3h$, so the area of the end is $\frac{1}{2}(3h)h = \frac{3}{2}h^2$ and the volume of water is $v = \frac{3}{2}h^2(10\text{m}) = 15h^2\text{m}$. We are given $\frac{dv}{dt} = 12\text{m}^3/\text{min}$ and want $\frac{dh}{dt}$ when $h = 0.5\text{m}$. Differentiating with respect to t gives $\frac{dv}{dt} = 30h\frac{dh}{dt}\text{m}$, so

$$\frac{dh}{dt} = \frac{1}{30h\text{m}}\frac{dv}{dt} = \frac{1}{30/2\text{m}^2}12\text{m}^3/\text{min} = \frac{4}{5}\text{m/min}\,.$$

/5 6. (a) State the Squeeze Theorem. Identify its assumptions and its conclusions.

If (assumptions)

- $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and
- $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L\,,$$

then (conclusion)

$$\lim_{x\to a} g(x) = L\,.$$

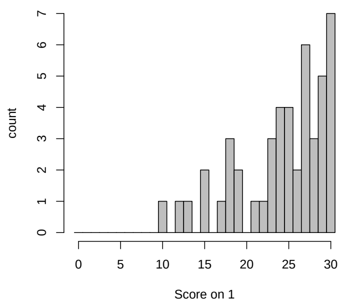
/5 (b) Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x\to 0}\left(5x^2\cos\left(\frac{1}{x}\right)+1\right)$.

Set

$$\begin{aligned} f(x) &= -5x^2 + 1\,,\\ g(x) &= 5x^2\cos\left(\frac{1}{x}\right) + 1\,, \quad \text{and}\\ h(x) &= 5x^2 + 1\,. \end{aligned}$$

Since $|\cos(\cdot)| \le 1$, we have $f(x) \le g(x) \le h(x)$ when x is near 0 (and for all $x \ne 0$). We can compute $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 1$, so the assumptions of the Squeeze theorem are satisfied and we can conclude $\lim_{x\to a} g(x) = 1$.

Scores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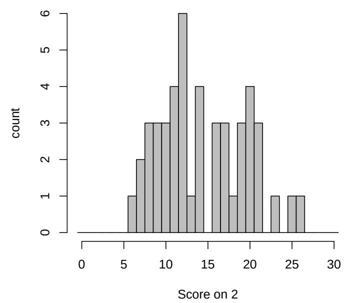
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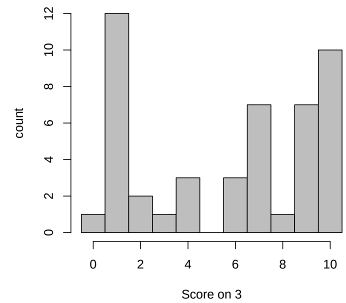
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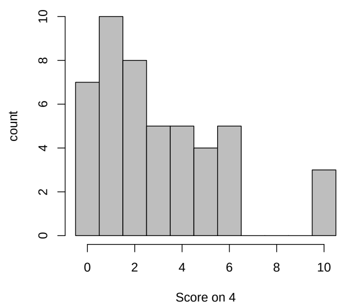
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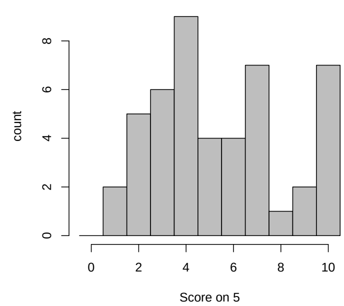
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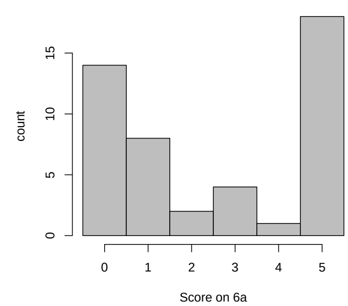
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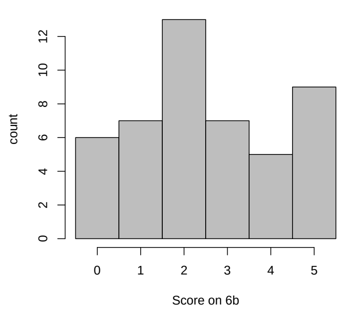
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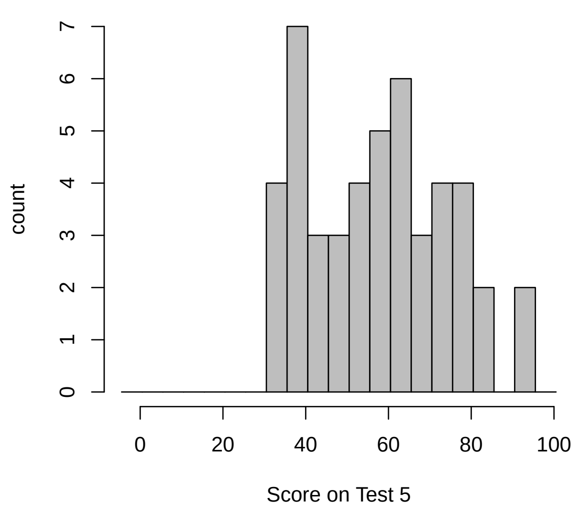
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Score on Test 5