score	possible	page
	20	1
	20	2
	30	3
	30	4
	100	

Name: ____________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/5 1. (a) Use the properties of exponents to simplify $\left(\frac{25}{4x^4y^5}\right)\left(\frac{5}{2x^3y^2}\right)^{-3}$.

$$= \left(\frac{25}{4x^4y^5}\right)\left(\frac{2x^3y^2}{5}\right)^3 \quad = \left(\frac{25}{4x^4y^5}\right)\left(\frac{2^3x^9y^6}{5^3}\right) \quad = \frac{25\cdot 2^3x^9y^6}{4x^4y^5\cdot 5^3} \quad = \frac{2x^5y}{5}\,.$$

/5 (b) Write the equation of the line passing through the two points $(1, 3)$ and $(3, 4)$.

We compute the slope by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{3-1} = \frac{1}{2}\,.$$

Using the point-slope form of a line $y - y_1 = m(x - x_1)$, we obtain $y - 3 = \frac{1}{2}(x - 1)$. (Optionally, we could then convert to slope intercept form and get $y = \frac{1}{2}x + \frac{5}{2}$.

/5 (c) Let $f(x) = 7x - 3$ and $g(x) = \frac{x+3}{7}$.

- Compute $(g \circ f)(x)$
- Compute $(f \circ g)(x)$
- Are f and g inverses of each other?

$$\begin{aligned}(g\circ f)(x) &= g(f(x)) = \frac{(7x-3)+3}{7} = \frac{7x}{7} = x\\ (f\circ g)(x) &= f(g(x)) = 7\left(\frac{x+3}{7}\right) - 3 = x + 3 - 3 = x\,.\end{aligned}$$

Since $(g \circ f)(x) = (f \circ g)(x) = x$, yes they are inverses.

/5 (d) Use the properties of logarithms to write $f(x) = 2\ln(x - 3) + \log_e(y + 2) - \ln(z)$ as a single logarithm.

$$f(x) = \ln((x-3)^2) + \ln(y+2) - \ln(z) \quad = \ln((x-3)^2(y+2)) - \ln(z) = \ln\left(\frac{(x-3)^2(y+2)}{z}\right).$$

2. Consider the rational function

$$f(x) = \frac{x^2 + 4x + 3}{1 - x^2}.$$

/4 (a) Express the domain of f in interval notation.

$f(x) = \frac{(x+1)(x+3)}{(1+x)(1-x)} = \frac{x+3}{1-x}$ except that there is a hole at $x = -1$. Since we also divide by 0 at $x = 1$, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

/4 (b) Find the x and y intercepts of f.

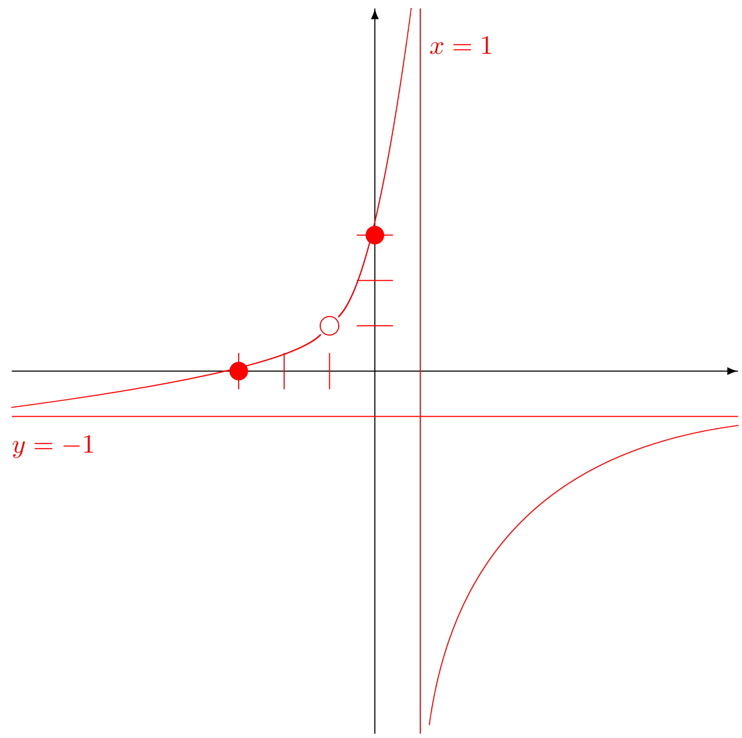
$f(0) = \frac{0+3}{1-0} = 3$ so the y-intercept is at $(0, 3)$. Setting $0 = \frac{x+3}{1-x}$ yields $x = -3$, so the x-intercept is at $(-3, 0)$.

/6 (c) Find all vertical and horizontal asymptotes and identify any holes.

Since $(1-x)$ remains in the denominator, $x = 1$ is a vertical asymptote. Horizontal asymptotes are determined by the highest powers in the numerator and denominator, so we have $\frac{x+3}{1-x} \to \frac{x}{-x} = -1$ and $y = -1$ is a horizontal asymptote.

As noted above, $x = -1$ gives a hole. The y-value is $\frac{-1+3}{1--1} = 1$.

/6 (d) Sketch a detailed graph of f.



/15 3. State the definition of "A function f is continuous at a number a". Let

$$f(x) = \begin{cases} \dfrac{2x^2 - x - 15}{x - 3} & \text{if } x < 3 \\ kx - 1 & \text{if } x \geq 3 \end{cases}.$$

Determine the value of k that will make the function f continous at 3, or explain why no value of k will work.

A function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

Since the function is piecewise defined with a change at a, we have to compute

$$\begin{aligned} f(3) &= 3k - 1\,, \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (kx - 1) &= 3k - 1\,, \qquad \text{and} \\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{2x^2 - x - 15}{x - 3} &= \lim_{x\to 3^-} \frac{(x-3)(2x+5)}{x - 3} = \lim_{x\to 3^-} 2x + 5 = 11\,. \end{aligned}$$

For $\lim_{x\to 3} f(x)$ to exist we need $3k - 1 = 11$ so $k = 4$ and $\lim_{x\to 3} f(x) = 11 = f(3)$.

/15 4. State the Intermediate Value Theorem. Identify what are its assumptions (hypotheses) and what are its conclusions. Use the Intermediate Value Theorem to show that the equation $10^x = x^2$ has a solution.

If (hypotheses)

- f is continuous on $[a, b]$ and
- $f(a) < N < f(b)$ or $f(a) > N > f(b)$,

then (conclusions) there exists $c \in (a, b)$ such that $f(c) = N$.

Let $f(x) = x^2 - 10^x$, so we want to show a solution to $f(x) = 0$ exists. Since x and 10^x are both continuous, so is $f(x)$. Plugging in, we find

$$f(0) = 0 - 10^0 = -1 < 0 \quad \text{and}$$
$$f(-1) = 1 - 10^{-1} = 9/10 > 0\,.$$

So, by the Intermediate Value Theorem, there must exist $-1 < c < 0$ such that $f(c) = 0$.

5. Compute the following limits. If you use the Squeeze Theorem, then indicate the two functions that you are using to squeeze.

/10 (a) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} =$

$$\lim_{x\to 4} \frac{x(x-4)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}\,.$$

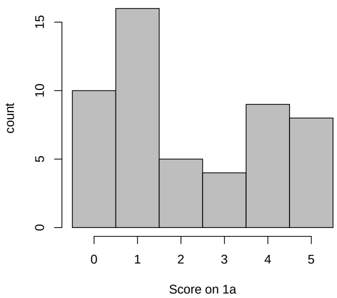
/10 (b) $\lim_{h\to 0} \frac{(2+h)^3-8}{h} =$

$$\lim_{h\to 0} \frac{(2^3+3(2^2h)+3(2h^2)+h^3)-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(12+6h+h^2)}{h}$$
$$= \lim_{h\to 0} (12+6h+h^2) = 12+0+0 = 12\,.$$

/10 (c) For $f(x) = (3x-1)^{-1}$, compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$

$$\lim_{h\to 0} \frac{(3(x+h)-1)^{-1}-(3x-1)^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{(3(x+h)-1)} - \frac{1}{(3x-1)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{(3x-1)-(3(x+h)-1)}{(3x-1)(3(x+h)-1)}}{h} = \lim_{h\to 0} \frac{-3h}{h(3x-1)(3(x+h)-1)}$$
$$= \lim_{h\to 0} \frac{-3}{(3x-1)(3(x+h)-1)} = \frac{-3}{(3x-1)(3(x+0)-1)} = \frac{-3}{(3x-1)^2}\,.$$

Scores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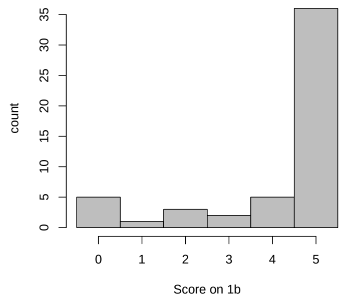
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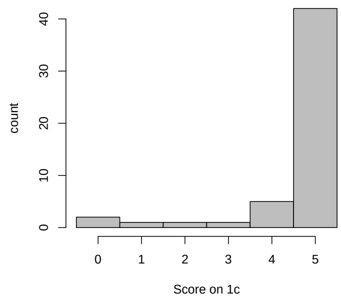
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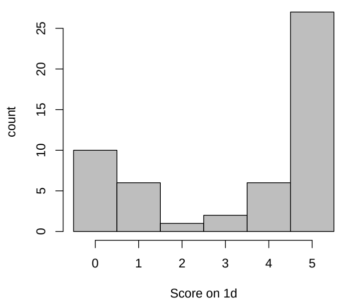
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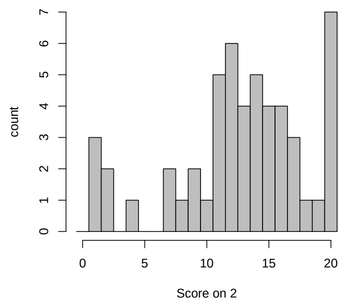
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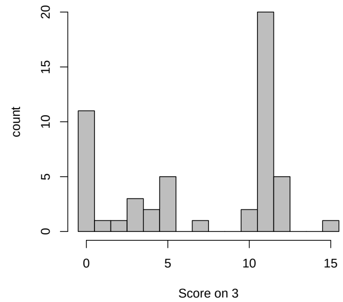
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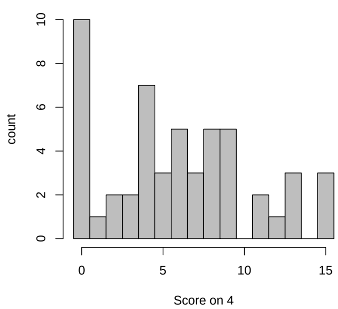
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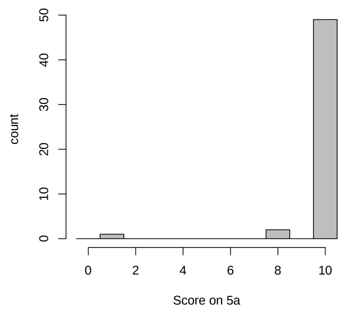
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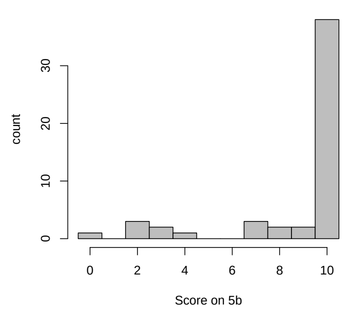
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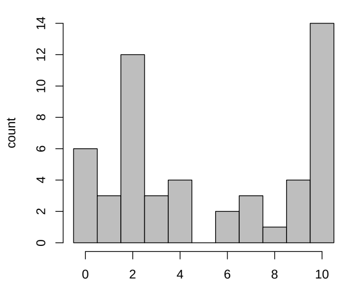
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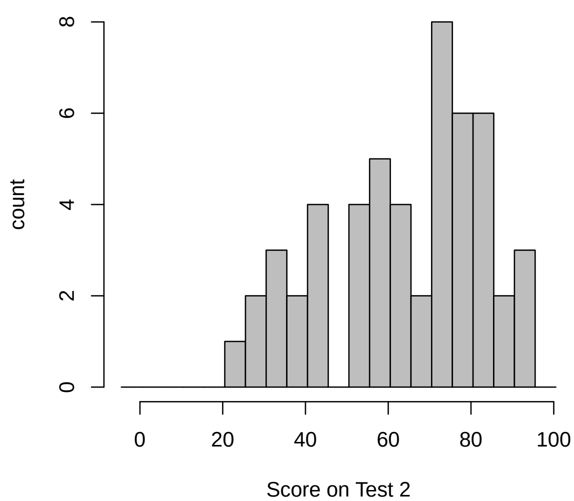
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Score on Test 2