Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student’s solutions. (You may ask me questions.)

1. (a) Use the properties of exponents to simplify \( \left( \frac{25}{4x^4y^5} \right) \left( \frac{5}{2x^3y^2} \right)^{-3} \).

\[
= \left( \frac{25}{4x^4y^5} \right) \left( \frac{2x^3y^2}{5} \right)^3
= \left( \frac{25}{4x^4y^5} \right) \left( \frac{2^3x^9y^6}{5^3} \right)
= \frac{25 \cdot 2^3x^9y^6}{4x^4y^5 \cdot 5^3}
= \frac{2x^5y}{5}.
\]

(b) Write the equation of the line passing through the two points (1, 3) and (3, 4).

We compute the slope by

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3}{3 - 1} = \frac{1}{2}.
\]

Using the point-slope form of a line \( y - y_1 = m(x - x_1) \), we obtain \( y - 3 = \frac{1}{2}(x - 1) \). (Optionally, we could then convert to slope intercept form and get \( y = \frac{1}{2}x + \frac{5}{2} \).

(c) Let \( f(x) = 7x - 3 \) and \( g(x) = \frac{x + 3}{7} \).

- Compute \((g \circ f)(x)\)
- Compute \((f \circ g)(x)\)
- Are \( f \) and \( g \) inverses of each other?

\[
(g \circ f)(x) = g(f(x)) = \frac{(7x - 3) + 3}{7} = \frac{7x}{7} = x
\]

\[
(f \circ g)(x) = f(g(x)) = 7 \left( \frac{x + 3}{7} \right) - 3 = x + 3 - 3 = x.
\]

Since \((g \circ f)(x) = (f \circ g)(x) = x\), yes they are inverses.

(d) Use the properties of logarithms to write \( f(x) = 2 \ln(x - 3) + \log_e(y + 2) - \ln(z) \) as a single logarithm.

\[
f(x) = \ln((x - 3)^2) + \ln(y + 2) - \ln(z) = \ln((x - 3)^2(y + 2)) - \ln(z) = \ln \left( \frac{(x - 3)^2(y + 2)}{z} \right).
\]
2. Consider the rational function

\[ f(x) = \frac{x^2 + 4x + 3}{1 - x^2}. \]

(a) Express the domain of \( f \) in interval notation.

\[ f(x) = \frac{(x+1)(x+3)}{(1+x)(1-x)} = \frac{x+3}{1-x} \] except that there is a hole at \( x = -1 \). Since we also divide by 0 at \( x = 1 \), the domain is \( (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \).

(b) Find the \( x \) and \( y \) intercepts of \( f \).

\[ f(0) = \frac{0+3}{1} = 3 \] so the \( y \)-intercept is at \( (0, 3) \). Setting \( 0 = \frac{x+3}{1-x} \) yields \( x = -3 \), so the \( x \)-intercept is at \( (-3, 0) \).

(c) Find all vertical and horizontal asymptotes and identify any holes.

Since \( (1-x) \) remains in the denominator, \( x = 1 \) is a vertical asymptote. Horizontal asymptotes are determined by the highest powers in the numerator and denominator, so we have \( \frac{x+3}{1-x} \to \frac{x}{-x} = -1 \) and \( y = -1 \) is a horizontal asymptote.

As noted above, \( x = -1 \) gives a hole. The \( y \)-value is \( \frac{-1+3}{1-(-1)} = 1 \).

(d) Sketch a detailed graph of \( f \).
3. State the definition of “A function \( f \) is continuous at a number \( a \)”. Let

\[
 f(x) = \begin{cases} 
 2x^2 - x - 15 & \text{if } x < 3 \\
 x - 3 & \text{if } x \geq 3 \\
 kx - 1 & \text{if } x \geq 3
\end{cases}
\]

Determine the value of \( k \) that will make the function \( f \) continuous at 3, or explain why no value of \( k \) will work.

A function \( f \) is continuous at a number \( a \) if \( \lim_{x \to a} f(x) = f(a) \).

Since the function is piecewise defined with a change at 3, we have to compute

\[
 f(3) = 3k - 1, \\
 \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (kx - 1) = 3k - 1, \\
 \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{2x^2 - x - 15}{x - 3} = \lim_{x \to 3^-} \frac{(x - 3)(2x + 5)}{x - 3} = \lim_{x \to 3^-} 2x + 5 = 11.
\]

For \( \lim_{x \to 3} f(x) \) to exist we need \( 3k - 1 = 11 \) so \( k = 4 \) and \( \lim_{x \to 3} f(x) = 11 = f(3) \).

4. State the Intermediate Value Theorem. Identify what are its assumptions (hypotheses) and what are its conclusions. Use the Intermediate Value Theorem to show that the equation \( 10^x = x^2 \) has a solution.

If (hypotheses)

- \( f \) is continuous on \([a, b]\) and
- \( f(a) < N < f(b) \) or \( f(a) > N > f(b) \),

then (conclusions) there exists \( c \in (a, b) \) such that \( f(c) = N \).

Let \( f(x) = x^2 - 10^x \), so we want to show a solution to \( f(x) = 0 \) exists. Since \( x \) and \( 10^x \) are both continuous, so is \( f(x) \). Plugging in, we find

\[
 f(0) = 0 - 10^0 = -1 < 0 \\
 f(-1) = 1 - 10^{-1} = 9/10 > 0.
\]

So, by the Intermediate Value Theorem, there must exist \(-1 < c < 0\) such that \( f(c) = 0 \).
5. Compute the following limits. If you use the Squeeze Theorem, then indicate the two functions that you are using to squeeze.

(a) \[ \lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \]

\[ \lim_{x \to 4} \frac{x(x - 4)}{(x - 4)(x + 1)} = \lim_{x \to 4} \frac{x}{x + 1} = \frac{4}{5} = \frac{4}{4 + 1} = \frac{4}{5}. \]

(b) \[ \lim_{h \to 0} \frac{(2 + h)^3 - 8}{h} = \]

\[ \lim_{h \to 0} \frac{2^3 + 3(2^2h) + 3(2h^2) + h^3 - 8}{h} = \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(12 + 6h + h^2)}{h} = 12 + 0 + 0 = 12. \]

(c) For \( f(x) = (3x - 1)^{-1}, \) compute \[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \]

\[ \lim_{h \to 0} \frac{(3(x + h) - 1) - (3x - 1)^{-1}}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{3(x+h) - 1}{(3x-1)(3(x+h) - 1)} \right) = \lim_{h \to 0} \frac{-3h}{h(3x - 1)(3(x + h) - 1)} = \frac{-3}{(3x - 1)(3x + 0 - 1)} = \frac{-3}{(3x - 1)^2}. \]