1. Sketch the graph of a single function that has all of the following properties:

(a) Continuous everywhere.

(b) \( f'(x) > 0 \) if \( |x| < 2 \).

(c) \( f'(x) < 0 \) if \( |x| > 2 \).

(d) \( f'(-2) = 0 \).

(e) \( f \) is not differentiable at \( x = 2 \).

(f) \( \lim_{x \to 2} |f'(x)| = \infty \).

(g) \( f''(x) > 0 \) if \( x \neq 2 \).

(h) \( f(2) = 3 \).

Organizing into a chart, we have:

\[
\begin{array}{cccccc}
 f'' & + & + & + & \text{DNE} & + \\
 f' & + & + & 0 & + & \text{DNE} & - \\
 \hline
 (\infty, -2) & -2 & (-2, 2) & 2 & (2, \infty) \\
\end{array}
\]

There is a minimum at \( x = -2 \) and a cusp at \( x = 2 \).
2. Let \( f(x) = 2x^3 - 3x^2 - 12x + 3 \)

(a) Find the intervals where \( f \) is increasing, and the intervals where it is decreasing.

\[
f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)
\]

so the critical numbers are \( x = 2 \) and \( x = -1 \). The sign chart is

\[
\begin{array}{c|c|c|c}
  & (-\infty, -1) & (-1, 2) & (2, \infty) \\
\hline
f' & + & 0 & - & + \\
f & \nearrow & \searrow & \nearrow & \\
\end{array}
\]

so \( f \) is increasing on \((-\infty, -1)\) and \((2, \infty)\) and decreasing on \((-1, 2)\).

(b) Find the intervals where \( f \) is concave up, and the intervals where it is concave down.

\[
f''(x) = 6(2x - 1)
\]

so \( f''(x) = 0 \) at \( x = 1/2 \). The sign chart is

\[
\begin{array}{c|c|c}
  & (-\infty, 1/2) & (1/2, \infty) \\
\hline
f'' & - & + \\
f & \nearrow & \searrow & \nearrow \\
\end{array}
\]

so \( f \) is concave up on \((1/2, \infty)\) and concave down on \((-\infty, -1)\).

(c) Find the absolute maximum and minimum values of \( f \) on the interval \([-2, 0]\).

The only critical number in the interval is \( x = -1 \). Evaluating there and at the endpoints we get

\[
\begin{align*}
f(-2) &= -16 - 12 + 24 + 3 = -1, \\
f(-1) &= -2 - 3 + 12 + 3 = 10, \quad \text{and} \\
f(0) &= 3.
\end{align*}
\]

Thus the absolute maximum is 10 and occurs at \( x = -1 \) and the absolute minimum is -1 and occurs at \( x = -2 \).
3. For the function \( f(x) = \frac{x}{x^2 - 9} \)

(a) Find the x- and y-intercepts.

(b) Find any asymptotes.

(c) Find the intervals on which \( f \) is increasing or decreasing.

(d) Find the local maximum and minimum values of \( f \).

(e) Find the intervals of concavity and the inflection points.

(f) Use the information above to sketch the graph.

\( f(0) = 0 \) and no other \( x \) makes \( f(x) = 0 \), so both intercepts are at \((0, 0)\).

The denominator is 0 and there are vertical asymptotes at \( x = -3 \) and \( x = 3 \).

\[
\lim_{x \to \pm \infty} \frac{x}{x^2 - 9} = \lim_{x \to \pm \infty} \frac{x}{x^2} = \lim_{x \to \pm \infty} \frac{1}{x} = 0
\]

so there is a horizontal asymptote at \( y = 0 \).

\[
f'(x) = \frac{1(x^2 - 9) - x(2x)}{(x^2 - 9)^2} = -\frac{x^2 - 9}{(x^2 - 9)^2},
\]

which is undefined at \( x = \pm 3 \) but is never 0.

\[
f''(x) = -\frac{2x(x^2 - 9)^2 - (x^2 - 9)(x^2 - 9)2x}{(x^2 - 9)^4} = -2x(x^2 - 9) - (x^2 - 9)2x = 2x(x^2 + 9 + 2x^2 + 18) = \frac{2x(x^2 + 27)}{(x^2 - 9)^3},
\]

which is undefined at \( x = \pm 3 \) and 0 at \( x = 0 \).

Assembling into a chart and checking signs, we have

\[
\begin{array}{cccc}
\text{I.P.} & \text{V.A.} & \text{I.P.} & \text{V.A.} \\
\hline
f'' & - & DNE & + & 0 & - & DNE & + \\
f' & - & DNE & - & - & - & DNE & - \\
\hline
\end{array}
\]

The are no local maxima or minima. There is an inflection point at \((0, 0)\).
4. (a) State the Mean Value Theorem (MVT).

If • $f$ is continuous on the closed interval $[a, b]$ and
• $f$ is differentiable on the open interval $(a, b)$,
then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

(b) State why the function

$$f(x) = x^3 - 3x + 2$$
on the interval $[-2, 2]$
satisfies each of the hypotheses of the MVT on the given interval. Then find all numbers $c$ that satisfy the conclusion of the MVT.

Since $f$ is a polynomial, it is continuous and differentiable everywhere, and so satisfies the hypotheses.

We want $c \in (-2, 2)$ so that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{8 - 6 + 2 - (-8 + 6 + 2)}{2 - (-2)} = 1.$$ 

We have $f'(x) = 3x^2 - 3$ so we set up the equation $3x^2 - 3 = 1$, which has solutions $c = \pm \sqrt{4/3}$, both of which are in $(-2, 2)$.

5. Compute the following:

(a) $\frac{d}{dx}[x^x] =$

Set $y = x^x$ so $\ln(y) = x \ln(x)$. Differentiating yields $\frac{y'}{y} = 1 + \frac{1}{x} = \ln(x) + 1$, so $y' = x^x (\ln(x) + 1)$.

(b) $\lim_{x \to \infty} x^x = \infty \infty = \infty$

(c) $\lim_{x \to 0^+} x^x =$

Directly plugging in gives the indeterminate form $0^0$. Instead we use properties of exponentials and logarithms and continuity to transform to

$$\lim_{x \to 0^+} e^{x \ln(x)} = e^{\lim_{x \to 0^+} x \ln(x)}.$$ 

Directly evaluating the inner limit gives indeterminate form $0(-\infty)$. Rewriting to get $\infty/\infty$ form, we can apply L’Hôpital’s rule to get

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} -x = 0.$$ 

The original limit is thus $e^0 = 1$. 