Show your work!

You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student’s solutions. (You may ask me questions.)

1. For the function \( f(x) = (2x + 1)^{-1} \), the tangent line at \( x = 2 \) has equation \( y = \left(-\frac{2}{25}\right)(x - 2) + \frac{1}{5} \).

Graph \( f(x) \) and the tangent line.

There is a vertical asymptote at \( x = -\frac{1}{2} \) and a horizontal asymptote at \( y = 0 \). Near the vertical asymptote, \( \lim_{x \to (-1/2)^+} f(x) = \infty \) and \( \lim_{x \to (-1/2)^-} f(x) = -\infty \). The tangent line has \( y \)-intercept at \( \frac{9}{25} \).

2. State

- the definition of “Continuous” and
- the definition of “Differentiable”.

Give an example of a function that is one but not the other.

- A function \( f \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \).

- A function \( f \) is differentiable at \( a \) if \( f'(a) \) exists.

The function \( f(x) = |x| \) is continuous (everywhere), but is not differentiable at \( a = 0 \) since

\[
\lim_{h \to 0^-} \frac{|0 + h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1 \neq \lim_{h \to 0^+} \frac{|0 + h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.
\]
3. Compute the following derivatives:

/10 (a) \( f(x) = \sqrt{x^2 + \frac{3}{x} + x^{3/4} + \cot(x) - \sin(7)} \)
\( \Rightarrow f'(x) = \frac{2x - 3x^{-2} + \frac{3}{4}x^{-1/4} - \csc^2(x) - 0}{2\sqrt{x^2 + \frac{3}{x} + x^{3/4} + \cot(x) - \sin(7)}} \)

/10 (b) \( D_x \left[ \cos(x) \sin(8 + x^5 + 3x) \right] = -\sin(x) \sin(8 + x^5 + 3x) + \cos(x) \cos(8 + x^5 + 3x)(0 + 5x^4 + 3) \)

/10 (c) \( y = \tan^5(6x^3 - 7x) \Rightarrow \frac{dy}{dx} = 5 \tan^4(6x^3 - 7x) \sec^2(6x^3 - 7x)(18x^2 - 7) \)
4. Use implicit differentiation to find an equation for the tangent line to the curve defined by \( y^3 + x^2y^4 = 1 + 2x \) at the point \((0,1)\).

Differentiating both sides with respect to \(x\) yields

\[
3y^2 \frac{dy}{dx} + 2xy^4 + 4x^2y^3 \frac{dy}{dx} = 0 + 2.
\]

Solving for \(\frac{dy}{dx}\) yields

\[
\frac{dy}{dx} = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}.
\]

At \((0,1)\) this yields slope \(2/3\) and so the tangent line is

\[
y = \frac{2}{3}(x - 0) + 1.
\]

(We should also check that \((0,1)\) is on the curve by plugging in the original equation to get \(1+0 = 1+0\).)

5. State the Intermediate Value Theorem. Identify what are its assumptions (hypotheses) and what are its conclusions.

If (hypotheses)

- \(f\) is continuous on \([a, b]\) and
- \(f(a) < N < f(b)\) or \(f(a) > N > f(b)\),

then (conclusions) there exists \(c \in (a, b)\) such that \(f(c) = N\).
6. A trough is 10m long and its ends have the shape of isosceles triangles that are 3m across at the top and have a height of 1m. The trough is being filled with water at a rate of 12m³/min. Draw and label a diagram illustrating this scenario. How fast is the water level rising when it is 0.5m deep?

Since the triangular end has proportion 3/1, for a given height of water $h$ the width is $3h$, so the area of the end is $\frac{3}{2}h^2$ and the volume of water is $v = 15h^2$. We are given $\frac{dv}{dt} = 12m^3/min$ and want $\frac{dh}{dt}$ when $h = 0.5$. Differentiating with respect to $t$ gives $\frac{dv}{dt} = 30h \frac{dh}{dt}$, so

$$\frac{dh}{dt} = \frac{1}{30h} \frac{dv}{dt} = \frac{1}{30/2m^2} 12m^3/min = \frac{4}{5} m/min.$$

7. Use a linear approximation (or differentials) to estimate $(8.03)^{2/3}$.

Set $f(x) = x^{2/3}$ so $f'(x) = (2/3)x^{-1/3}$. Selecting $a = 8$ we have the linear approximation

$$f(x) \approx L_8(x) = f(8) + f'(8)(x - 8) = 4 + \frac{1}{3}(x - 8)$$

so $(8.03)^{2/3} = f(8.03) \approx 4 + \frac{0.03}{3} = 4.01$. 