1. Solve the equation and describe all solutions that lie in the interval $[0, 2\pi)$:

$$2 \sin^2(\theta) = 5 \sin(\theta) + 3.$$

$$\iff 0 = 2 \sin^2(\theta) - 5 \sin(\theta) - 3$$

$$\iff 0 = (2 \sin(\theta) + 1)(\sin(\theta) - 3)$$

$$\iff \sin(\theta) = -1/2 \text{ or } \sin(\theta) = 3.$$  

The first equation has solutions $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The second equation has no solutions.

2. Use the properties of logarithms to rewrite the following expressions:

- Write as a single logarithm:
  $$\log_2(x^3 - 4) + 2 \log_2(x + 2) - 4 \log_2(3x + 2)$$

- Write so that the result does not contain any powers, products, or quotients:

  $$\log_3 \left( \frac{xy}{z^2 \sqrt{w}} \right)$$

  $$\log_2 \left( \frac{(x^3 - 4)(x + 2)^2}{(3x + 2)^4} \right)$$

  $$\log_3(x) + \log_3(y) - 2 \log_3(z) - \frac{1}{3} \log_3(w)$$

Show your work!

You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student’s solutions. (You may ask me questions.)
3. Consider the rational function

\[ f(x) = \frac{x^2 + 4x + 3}{x^2 - 1}. \]

(a) Express the domain of \( f \) in interval notation.
(b) Find the \( x \) and \( y \) intercepts of \( f \).
(c) Find all vertical and horizontal asymptotes.
(d) Identify any holes.
(e) Sketch a detailed graph of \( f \).

\[
\begin{align*}
  f(x) &= \frac{(x+1)(x+3)}{(x+1)(x-1)} = \frac{x+3}{x-1} \text{ except that there is a hole at } x = -1. \\
  \text{Since we also divide by 0 at } x = 1, \text{ the domain is } (-\infty, -1) \cup (-1, 1) \cup (1, \infty).
\end{align*}
\]

\[
\begin{align*}
  f(0) &= \frac{0+3}{0-1} = -3 \text{ so the } y\text{-intercept is at } (0, -3). \text{ Setting } 0 = \frac{x+3}{x-1} \text{ yields } x = -3, \text{ so the } x\text{-intercept is at } (-3, 0).
\end{align*}
\]

Since \( (x-1) \) remains in the denominator, \( x = 1 \) is a vertical asymptote. Horizontal asymptotes are determined by the highest powers in the numerator and denominator, so we have \( \frac{x+3}{x-1} \to \frac{x}{x} = 1 \) and \( y = 1 \) is a horizontal asymptote.

As noted above, \( x = -1 \) gives a hole. The \( y\)-value is \( \frac{-1+3}{-1-1} = -1 \).

\[
\begin{align*}
  y &= 1 \quad \text{at } x = 1. \\
  x &= 1 \quad \text{at } y = 1.
\end{align*}
\]
4. Let \( g(x) = \begin{cases} 
\frac{2x^2 - x - 15}{x - 3} & \text{if } x \neq 3 \\
kx - 1 & \text{if } x = 3 
\end{cases} \). Determine the value of \( k \) that will make the function \( g \) continuous, or explain why no value of \( k \) will work.

For \( x \neq 3 \) we can reduce to
\[
\frac{2x^2 - x - 15}{x - 3} = \frac{(x - 3)(2x + 5)}{x - 3} = 2x + 5.
\]

From this we know that \( g \) is continuous when \( x \neq 3 \) and \( \lim_{x \to 3} = 2(3) + 5 = 11 \). To make \( g \) also continuous at \( x = 3 \), we need \( g(3) = k(3) - 1 = 11 \), which means \( k = 4 \).

5. Use the Intermediate Value Theorem to show that the equation \( 2^{-x} = x \) has a solution.

Let \( f(x) = x - 2^{-x} \), so we want to show a solution to \( f(x) = 0 \) exists. Since \( x \) and \( 2^{-x} \) are both continuous, so is \( f(x) \). Plugging in, we find
\[
\begin{align*}
    f(0) &= 0 - 2^0 = -1 < 0 \quad \text{and} \\
    f(1) &= 1 - 2^{-1} = 1/2 > 0.
\end{align*}
\]

So, by the Intermediate Value Theorem, there must exist \( 0 < c < 1 \) such that \( f(c) = 0 \).
6. Compute the following limits. If you use the Squeeze Theorem, then indicate the two functions that you are using to squeeze.

/10 (a) For \( f(x) = (2x + 1)^{-1} \), compute \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \)

\[
= \lim_{h \to 0} \frac{(2(x + h) + 1)^{-1} - (2x + 1)^{-1}}{h} \\
= \lim_{h \to 0} \frac{1}{h} \frac{1}{(2x+h)+1} - \frac{1}{(2x+1)} \\
= \lim_{h \to 0} \frac{(2x+1) - (2x+h+1)}{(2x+1)(2(x+h)+1)} \\
= \lim_{h \to 0} \frac{-2h}{h(2x+1)(2(x+h)+1)} \\
= \lim_{h \to 0} \frac{-2}{(2x+1)(2x+0+1)} = \frac{-2}{(2x+1)^2}.
\]

/10 (b) \( \lim_{x \to 0} 4x \cos(3/x) = \)

Since the range of \( \cos \) is \([-1,1]\), we have

\[ |4x \cos(3/x)| \leq 4|x| \text{ and so } -4|x| \leq 4x \cos(3/x) \leq 4|x|. \]

We have \( \lim_{x \to 0} -4|x| = 0 = \lim_{x \to 0} 4|x| \). So by the squeeze theorem, using the two functions \(-4|x|\) and \(4|x|\) to squeeze, we know \( \lim_{x \to 0} 4x \cos(3/x) = 0. \)

/10 (c) \( \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \)

\[
= \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\
= \lim_{x \to 9} \frac{(x - 9)(\sqrt{x} + 3)}{x - 9} \\
= \lim_{x \to 9} (\sqrt{x} + 3) \\
= \sqrt{9} + 3 = 6. 
\]