Here are some sample questions, some of which are from old tests. No books or notes are allowed.

1. Norms:

   (a) State the defining properties of a norm.

   (b) Let $f(x)$ be the function defined on vectors $x \in \mathbb{C}^m$ by ****. Is f a norm?

   (c) Give an example of a function defined on vectors $x \in \mathbb{C}^m$ that fails one of the properties of a norm but satisfies the others.

   (d) Define what it means for a matrix norm to be induced by a vector norm.

   (e) Show that $\|AB\| \leq \|A\| \, \|B\|$ if the matrix norm is induced by a vector norm.

   (f) For the matrix $A = [*\text{at most } 3 \times 3*]$ compute the norms $\|A\|_1$, $\|A\|_\infty$, and $\|A\|_F$.

2. SVDs:

   (a) State the formal definition of the Singular Value Decomposition for a matrix $A \in \mathbb{C}^{m \times n}$.

   (b) Prove the existence of an SVD for an arbitrary $A \in \mathbb{C}^{m \times n}$.

   (c) Explain under what conditions and in what sense the SVD is unique.

   (d) Prove: The rank of A is r, the number of nonzero singular values.

   (e) Prove: (some other properties from section 5)

   (f) Prove: For any ν with $0 \leq \nu < r$, if

   $$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

   with σ_j, u_j, and v_j from the SVD of A then

   $$\|A - A_\nu\|_2 = \inf_{\substack{B \in \mathbb{C}^{m \times n} \\ \text{rank}(B) \leq \nu}} \|A - B\|_2 = \sigma_{\nu+1} \, .$$